Horizon Boundary Mechanics: Force, Pressure, and Work from the Einstein Boundary Potential

J. T. Tyler

La Jolla, California, USA

Abstract

We formulate horizon boundary mechanics for the Einstein energy-valued boundary potential, resolving its area and normal-rate dependence into surface stress and acceleration-length work. With the signed length $L_s = c^2/\kappa$, the full differential is $dU_\partial = \Pi_\partial dA - F_L dL_s$, where $F_L = A\kappa^2/(8\pi G)$ and $P_L = F_L/A$. This mechanical construction carries over wherever the specified potential and normal-rate prescription apply; its canonical and geometric realizations retain their boundary conditions. In the selected nonexpanding Einstein null sector, the symplectic potential supplies the force as an action response, independently reproduced by the stationary area–boost corner pair. Rindler fixes the length normalization operationally, and Schwarzschild gives an exact Smarr representation. As a major cosmological application, the separately calibrated Cai–Kim FRW screen gives $P_L = \rho/3$, pressure–volume work, and a pressure organization of the Friedmann system using the standard bulk-energy identification. Thermal translation follows the mechanical derivation. The framework concerns a boundary potential, not a new bulk stress tensor or field equation.

Keywords: horizon boundary mechanics, gravitational boundary response, acceleration length, null symplectic structure, FRW cosmology

1. Horizon boundary mechanics

Horizon boundary mechanics studies how a specified gravitational boundary potential responds to changes of cut area and normal scale. Area and surface gravity already enter black-hole mechanics, membrane stress, and the gravitational surface Hamiltonian [1, 2, 3, 5]. The common potential is

$$U_\partial(A, \kappa) = \frac{Ac^2\kappa}{8\pi G}. \qquad (1)$$

Here $A = \int_S \epsilon_S$ is the cut's intrinsic area; a spherical radius is not assumed. The potential is energy-valued, not universally the total spacetime energy: $U_\partial = Mc^2/2$ for standard Schwarzschild, the Misner–Sharp horizon energy in the Cai–Kim FRW realization, and a finite-patch potential without enclosed source mass in Rindler.

The normal-rate direction becomes force–displacement work when expressed through acceleration length. The construction is not restricted to one horizon geometry: it applies to the stated potential with its physical flow, clock, and orientation prescribed. Canonical realization is a separate question about boundary data. The null response is the primary action-level derivation; the stationary corner pair supplies an independent check of the same coefficient, not an extra degree of freedom. Other boundary choices can define different charges and responses.

The work term precedes thermality. Rindler and Schwarzschild supply stationary realizations; FRW is a cosmological application, retaining its established bulk-energy relation and selected rate. This

Email address: jttyler101@gmail.com (J. T. Tyler)

gives the work coordinate a mechanical basis before imposing a thermal first law [6].

2. Mechanical response and boost interpretation

On a nonzero signed branch define

$$L_s = \frac{c^2}{\kappa}, \quad L_\kappa = |L_s|, \quad \omega_s = \frac{\kappa}{c}, \quad \Omega_\kappa = \frac{|\kappa|}{c}. \tag{2}$$

The positive length is the light-travel length associated with timescale $1/\Omega_\kappa$. Up to a dimensionless factor it is the only length constructible from c and κ without another scale. Rindler's exact proper distance c^2/a fixes the factor to one [7]. Other sectors need not make this reciprocal-rate scale a proper radial distance.

The potential and its derivatives are

$$U_\partial = \frac{Ac^4}{8\pi G L_s}, \qquad \Pi_\partial = \left(\frac{\partial U_\partial}{\partial A}\right)_{L_s} = \frac{c^2\kappa}{8\pi G}, \tag{3}$$

$$F_L = -\left(\frac{\partial U_\partial}{\partial L_s}\right)_A = \frac{A\kappa^2}{8\pi G}, \qquad \boxed{P_L = \frac{F_L}{A} = \frac{\kappa^2}{8\pi G}}. \tag{4}$$

Π_∂ has force-per-length units; P_L has force-per-area units. With $M_\partial = U_\partial/c^2$, the exact identity is $F_L = M_\partial\kappa$, not a separate equation of motion for a material body. The full differential is

$$\boxed{dU_\partial = \Pi_\partial\, dA - F_L\, dL_s}, \qquad \Pi_\partial = P_L L_s. \tag{5}$$

The established Einstein momentum conjugate to the normal boost angle is $J_\eta = Ac^3/(8\pi G)$ [11, 12]. Hence

$$\boxed{U_\partial = \omega_s J_\eta, \qquad U_\partial L_s = cJ_\eta.} \tag{6}$$

J_η has action units and is not ordinary rotational angular momentum. Multiplying this boost charge by the selected normal rate gives the boundary energy. The complete dictionary is purely mechanical:

$$\begin{aligned} dU_\partial &= \omega_s\, dJ_\eta + J_\eta\, d\omega_s, \\ \omega_s\, dJ_\eta &= \Pi_\partial\, dA, \qquad J_\eta\, d\omega_s = -F_L\, dL_s. \end{aligned} \tag{7}$$

The charge and rate contributions are distinct variations of the same potential. For fixed area, the product $U_\partial L_s/c$ is independent of the rate. Positive magnitudes obey $|U_\partial| L_\kappa = cJ_\eta$.

The primary work term needs no volume definition:

$$F_L\, dL_s = P_L \vartheta_V, \qquad \vartheta_V = A\, dL_s. \tag{8}$$

On independent (A, L_s) data, $d\vartheta_V = dA \wedge dL_s$ and $d(AL_s) = A\, dL_s + L_s\, dA$. At fixed area the form is exact; on a specified spherical family it can equal a geometric volume differential. A physical radius or displacement requires its own Jacobian. The common Euler identity is $U_\partial = \Pi_\partial A = F_L L_s$; it does not split the potential into independent stored energies.

3. Canonical foundation of the response

The gravitational action variation gives interior field equations and a boundary one-form Θ. This symplectic potential is linear in a field variation, like $p\,\delta q$; it is not the energy function U_∂. Its field-space exterior derivative is the canonical two-form. A choice of boundary terms determines the displayed source and response.

Let a null hypersurface $\mathcal{N}$ be foliated by spacelike two-dimensional cuts S, with metric q_{AB} and area element $\epsilon_S = \sqrt{q}\, d^2x$. For a selected null generator ξ^a, define

$$\nabla_\xi \xi^a = \widehat{\kappa}\xi^a, \qquad \theta = \frac{1}{\sqrt{q}}\mathcal{L}_\xi \sqrt{q}. \tag{9}$$

Inaffinity $\widehat{\kappa}$ measures nonaffine generator parameterization; expansion θ measures fractional area change. With a length parameter both have inverse-length units, and $\kappa = c^2\widehat{\kappa}$. A null generator has no proper time, so its inaffinity is not a material observer's proper acceleration.

Hopfmüller and Freidel identify the scalar canonical combination [8, 9]

$$\mu = \widehat{\kappa} + \frac{1}{2}\theta. \tag{10}$$

Shear and generator/twist data occupy other sectors. In units $c = 1$, the area polarization has scalar term $\Theta_A^{(0)} = -(8\pi G)^{-1} \int \mu\, \delta\epsilon_S\, dv$. Adding the exact variation of $B_\mu = (8\pi G)^{-1} \int \mu\epsilon_S\, dv$ gives

$$\Theta_\mu^{(0)} = \Theta_A^{(0)} + \delta B_\mu = \frac{1}{8\pi G} \int_N \epsilon_S\, \delta\mu\, dv. \tag{11}$$

This is $-p\delta q + \delta(pq) = q\delta p$: a change of polarization, leaving the two-form unchanged. Covariant mixed boundary conditions identify the same derivative-containing scalar source [10]. Orientation, physical generator, cut identifications, and other boundary/corner data are specified.

Restrict to the nonexpanding sector and variations tangent to it:

$$\theta = 0, \qquad \delta\theta = 0. \tag{12}$$

Restoring acceleration units and the prescribed physical time label τ gives

$$\Theta_\kappa^{(0)} = \frac{c^2}{8\pi G} \int d\tau \int_S \epsilon_S\, \delta\kappa. \tag{13}$$

For uniform stationary data, this per-time coefficient is exactly the rate leg of dU_∂. Since $\delta\kappa = -c^2 \delta L_s / L_s^2$,

$$\boxed{\Theta_L^{(0)} = -\int d\tau \int_S \epsilon_S\, P_L \delta L_s = -\int d\tau F_L \delta L_s.} \tag{14}$$

The last equality assumes a uniform finite cut. Once the canonical rate response is supplied, this substitution is direct. It neither adds a thermal ansatz nor requires another dynamical equation.

An admissible source variation compares nearby boundary-value problems while controlling the remaining data. Within one conservative problem the prescribed mixed source is fixed. The variation must preserve the stated nullness and nonexpanding conditions and account for matter and other symplectic responses. Fixing the clock prescription does not impose $\delta\kappa = 0$: the geometry can change with that prescription held constant.

Equivalently, the mixed on-shell action has local Hamilton–Jacobi response

$$F_L(\tau) = -\frac{\delta I_{\mathrm{os}}}{\delta L_s(\tau)}. \tag{15}$$

The time measure makes this derivative a force. A constant-source derivative across a fixed interval instead equals $-\Delta\tau F_L$. This response depends on the specified action and counterterm prescription; arbitrary source-dependent additions need not preserve it. No existence theorem for every freely assigned boundary source is claimed.

In this scalar sector,

$$\frac{\mathbf{\Omega}^{(0)}}{\Delta\tau} = \delta A \wedge \delta\Pi_\partial = \delta L_s \wedge \delta F_L. \tag{16}$$

These are representations of one pair. The Legendre term selects rate polarization; $\kappa \mapsto c^2/\kappa$ is then an invertible coordinate change, not a second Legendre transform. Algebra permits other coordinates. The independent length, Rindler, force, and work criteria select the mechanical interpretation used here.

3.1. Stationary area–boost check

A corner is the codimension-two intersection of boundary pieces. Its normal frames have relative Lorentz boost angle η, with one-form $J_\eta \delta\eta$ [11, 12]. For a stationary interval,

$$\eta = \frac{\kappa\Delta\tau}{c} = \frac{c\Delta\tau}{L_s}. \tag{17}$$

Changing variables gives

$$\begin{aligned} J_\eta \delta\eta &= p_L \delta L_s, \\ p_L &= -\frac{c\Delta\tau J_\eta}{L_s^2} = -\Delta\tau F_L. \end{aligned} \tag{18}$$

p_L has momentum units; F_L is the response per unit boundary time. This independently reproduces the null coefficient in the common stationary setting. It does not identify a changing cosmological screen's elementary velocity rapidity with the horizon normal boost rate.

3.2. Scope of the certification

Stationary Rindler and Schwarzschild realize the stated construction, as does a consistently normalized stationary de Sitter horizon. A generic FRW apparent-horizon worldtube need not be null: one vanishing null expansion on each cut does not imply a nonexpanding tube. Flat dust FRW, for example, has $R_A = 3ct/2$ and horizon speed relative to comoving flow $c/2$; its tube is timelike.

On an expanding null boundary the scalar source is μ, not just $\widehat{\kappa}$. Rewriting its $\delta\kappa$ contribution retains an additional expansion contribution; it does not remove the other sectors. The FRW identities below instead use their explicit CK calibration. This is a distinction in canonical provenance, not a failure of their stated algebra.

4. Boundary potential versus mass first law

The usual mass first law is a Hamiltonian variation for a specified evolution, not the full differential of $A\kappa$. Covariant charge variations include a symplectic-potential subtraction and generator corrections when required. Consequently $d(TS) = T\,dS + S\,dT$ does not require adding $S\,dT$ to the mass law. Surface-Hamiltonian treatments already display the complementary $T\delta S$ and $S\delta T$ contributions [5, 4].

Where the positive-branch thermal identification applies, Eq. (7) becomes

$$\begin{aligned} \omega_s\, dJ_\eta &= \Pi_\partial\, dA = T\, dS, \\ J_\eta\, d\omega_s &= -F_L\, dL_\kappa = S\, dT. \end{aligned} \tag{19}$$

The scale leg is not generally a separate state energy: its exterior derivative is $(c^2/8\pi G)\delta A \wedge \delta\kappa$. The complete dU_∂ is exact. On a one-dimensional physical family the isolated work term is integrable along the path, without acquiring universal independence.

For Schwarzschild, $E = Mc^2 = 2U_\partial$ and the constrained family gives

$$dE = T\, dS = 2F_L\, dL_\kappa, \qquad dU_\partial = F_L\, dL_\kappa. \tag{20}$$

Appending the boundary scale term as an independent addition to $dE = T\,dS$ would double-count that variation.

5. Stationary and cosmological realizations

On a spherical marginal horizon, $A = 4\pi R^2$ and the Misner–Sharp energy is $E_{\rm MS} = c^4R/(2G)$ [15, 16]. The area potential therefore has the spherical form

$$\boxed{U_\partial = E_{\rm MS}\frac{R}{L_s}, \qquad U_\partial L_s = cJ_\eta = E_{\rm MS}R.}$$

For $\kappa > 0$, $L_s = L_\kappa$: the ratio $U_\partial/E_{\rm MS} = R/L_\kappa$ is one in Cai–Kim FRW and one half in standard Schwarzschild. This is a realization, not a replacement for the area potential. The horizon formula is not the energy of an arbitrary sphere, and $E_{\rm MS} \propto R$ must be retained in any differential.

5.1. Rindler and the local observer meaning

For an observer with constant proper acceleration a in Minkowski spacetime,

$$\kappa = a, \qquad L_\kappa = c^2/a, \qquad P_L = a^2/(8\pi G). \tag{21}$$

The length is exactly the observer–horizon proper distance on the instantaneous rest slice. A rectangular transverse patch has $A = \int dy\,dz = \Delta y\,\Delta z$, independently of L_κ. No enclosed source mass, spherical radius, or projection is required.

The FGP local energy for stationary observers at small proper distance ℓ from a nonextremal horizon has the leading form $E_{\rm FGP} \simeq Ac^4/(8\pi G\ell)$ with $L_{\rm loc} \simeq \ell$ [13]. Its negative fixed-area distance derivative already has the inverse-square coefficient. The fixed-distance local first law and a variation moving the observer are different protocols: the latter also changes local clock normalization. In Schwarzschild $L_{\rm loc} = N(r)L_\infty \simeq \ell$ near the horizon, while $L_\infty = 2R_H$. This is a close local antecedent, not a universal identification of proper distance with acceleration length.

5.2. Schwarzschild

With stationary Killing time normalized at infinity,

$$\kappa = \frac{c^2}{2R_H}, \quad L_\kappa = 2R_H, \quad F_L = \frac{c^4}{8G}, \quad U_\partial = \frac{Mc^2}{2}. \tag{22}$$

The Smarr relation becomes

$$\boxed{Mc^2 = TS + F_L L_\kappa, \qquad TS = F_L L_\kappa = \frac{Mc^2}{2}.} \tag{23}$$

The products coincide pointwise, not after time averaging. They represent different derivative directions of one potential, not independently stored kinetic and potential energies. Radius work has the exact Jacobian

$$F_L\, dL_\kappa = 2F_L\, dR_H = (M\kappa)\, dR_H = 2P_L\, dV_H. \quad (24)$$

The area leg remains. At the horizon the Newtonian field magnitude equals κ, so the familiar longitudinal field-stress magnitude agrees with P_L; this sectoral comparison does not make P_L a covariant bulk stress tensor.

5.3. Cosmological application: the FRW screen

For four-dimensional FRW with arbitrary spatial curvature,

$$R_A = \frac{c}{\sqrt{H^2 + kc^2/a^2}}, \qquad H = \dot{a}/a. \quad (25)$$

The geometric Misner–Sharp energy at a marginal sphere and its Einstein–FRW matter identification give [15, 21]

$$E_A = \frac{c^4 R_A}{2G} = \rho V_A, \qquad V_A = \frac{4\pi R_A^3}{3}. \quad (26)$$

ρ is total bulk energy density and p_m total bulk pressure in the same units. V_A is areal volume, not generally the proper spatial volume for $k \neq 0$. The relation $E_A = \rho V_A$ is the established Einstein matter–geometry input, not an independent result of the pressure definition.

The instantaneous Cai–Kim prescription [14] sets

$$\kappa_{\rm CK} = \frac{c^2}{R_A}, \qquad L_{\rm CK} = R_A. \quad (27)$$

Then the force, screen pressure, and integrated identity are

$$F_L^{\rm CK} = \frac{c^4}{2G}, \quad P_{\rm scr} = P_L^{\rm CK} = \frac{c^4}{8\pi G R_A^2} = \frac{\rho}{3}, \quad (28)$$

$$E_A = \rho V_A = U_\partial = TS = F_L L_{\rm CK} = 3P_{\rm scr} V_A. \quad (29)$$

The thermal equality uses the CK temperature parameter and Einstein entropy. Mechanically, the complete constrained differential is

$$dE_A = \Pi_{\rm CK}\, dA_A - P_{\rm scr}\, dV_A = T_{\rm CK}\, dS_A - P_{\rm scr}\, dV_A. \quad (30)$$

Along the spherical family, $dE_A/dV_A = P_{\rm scr}$. This total slope is not the unconstrained derivative $-(\partial E/\partial V)_S$ and not the cosmic material pressure.

Equation (30) is a screen-state law. Cai and Kim's matter-supply relation across a momentarily fixed horizon is $\delta Q_{\rm matt} = A_A(\rho + p_m)HR_A\, dt = T_{\rm CK} dS_A$ with orientation signs. One sorts change by boundary coordinate; the other sorts it by carrier. Exact mechanical state identities do not by themselves establish exact equilibrium thermality during arbitrary FRW evolution.

With $\kappa_E = 8\pi G/c^4$, the pressure representation is

$$\frac{1}{R_A^2} = \frac{H^2}{c^2} + \frac{k}{a^2} = \kappa_E P_{\rm scr}, \quad (31)$$

$${}^{(2)}\mathcal{R}_\perp \equiv \frac{2\ddot{a}}{ac^2} = -\kappa_E(P_{\rm scr} + p_m), \quad (32)$$

$$\dot{P}_{\rm scr} + H(3P_{\rm scr} + p_m) = 0. \quad (33)$$

$1/R_A^2$ is the horizon sphere's Gaussian curvature; ${}^{(2)}\mathcal{R}_\perp$ is the Ricci scalar of the normal time–radial metric, not the full spacetime Ricci scalar. Screen pressure sets the horizon scale, its sum with bulk pressure governs acceleration, and bulk enthalpy density $\rho + p_m = 3P_{\rm scr} + p_m$ governs its evolution. These reorganize, rather than alter, the Friedmann equations.

The signed Hayward cross-focusing rate is different [16, 17]:

$$\frac{\kappa_H^s}{\kappa_{\rm CK}} = \frac{p_m - P_{\rm scr}}{4P_{\rm scr}} = \frac{3w-1}{4}, \qquad w = p_m/\rho. \quad (34)$$

In radiation FRW it vanishes while $P_{\rm scr}$ remains finite; in de Sitter their magnitudes agree. The associated quadratic response is $P_L^H = (P_{\rm scr} - p_m)^2/(16P_{\rm scr})$. On a nonzero branch its scale work obeys

$$F_L^H\, dL_H^s = \frac{p_m - P_{\rm scr}}{4}\, dV_A - \frac{3}{4}E_A\, dw. \quad (35)$$

Changing rate prescription changes both potential and work allocation; it is not a sign change alone. Detailed dynamical and moving-boundary comparisons belong to the full treatment.

Kong's effective pressure is $P_{\mathrm{eff}} = -dE_A/dV_A = -P_{\mathrm{scr}}$ in four-dimensional Einstein FRW [18]. The present normal pressure is obtained before its FRW identification and realized in the stationary sectors above. The distinction concerns its variational origin and mechanical coordinate, not only the sign of a spherical slope.

6. Thermal translation and entropy density

A regular Hawking–Unruh/KMS state independently supplies the thermal normalization [7, 19, 20]. Smooth Euclidean normal rotation has period 2π, so

$$c\Delta\tau_E = 2\pi L_\kappa, \quad k_B T = \frac{\hbar\Omega_\kappa}{2\pi}, \quad \frac{S}{k_B} = \frac{2\pi J_\eta}{\hbar}. \tag{36}$$

The last identity uses Einstein entropy with the standard zero additive constant. On the positive branch,

$$dU_\partial = T\,dS - F_L\,dL_\kappa, \qquad \frac{S}{k_B} = \frac{2\pi U_\partial L_\kappa}{\hbar c}. \tag{37}$$

Thus $U_\partial L_\kappa/c$ has the established meaning of boost charge before its conversion to dimensionless entropy. The area coefficient divided by temperature gives

$$\frac{S}{A} = \frac{P_L L_\kappa}{T} = \frac{k_B}{4\ell_P^2}, \qquad \ell_P^2 = \frac{G\hbar}{c^3}. \tag{38}$$

The quarter arises from the Einstein $1/(8\pi G)$ and thermal $1/(2\pi)$ normalizations. No microscopic state count or more fundamental disk area follows from canceling factors in a spherical area formula.

7. Generality, scope, and conclusion

The area–rate potential, mixed null scalar source, area–boost pair, surface-Hamiltonian split, and thermal entropy normalization are established ingredients. The construction identifies acceleration length as the mechanical source coordinate that resolves the normal-rate contribution into a force, pressure, and work form. Once the selected canonical response is supplied, extraction of F_L and P_L is straightforward; physical selection of the coordinate and its sector dictionaries carry the interpretation.

The mechanics of the stated potential is common to its realizations; the canonical theorem retains the specified Einstein null sector, action polarization, physical clock, relational cuts, and remaining source data. At $\kappa = 0$, the regular rate replaces the singular reciprocal chart. Squaring removes a sign, not clock dependence. A general evolving FRW worldtube requires a separate canonical matching calculation; the exact CK screen-state identities stated here do not assert its completion.

Horizon boundary mechanics resolves one potential into transverse surface stress and normal force–displacement work. The null action establishes the rate response, the stationary corner pair checks it independently, and the boost dictionary explains the same mechanical structure. Rindler and Schwarzschild directly realize the canonical sector; Cai–Kim FRW gives its separately calibrated cosmological screen realization. The unifying object is the boundary potential, not a new bulk stress tensor or modification of Einstein's equations.

Declaration of generative AI and AI-assisted technologies

During preparation and revision, the author used OpenAI ChatGPT and Codex to assist with literature searches, algebraic checks, and presentation. The author takes full responsibility for the manuscript.

Data availability

No data were created or analyzed in this theoretical study.

References

[1] J. M. Bardeen, B. Carter, S. W. Hawking, The four laws of black hole mechanics, Commun. Math. Phys. 31 (1973) 161–170.

[2] L. Smarr, Mass formula for Kerr black holes, Phys. Rev. Lett. 30 (1973) 71–73.

[3] R. H. Price, K. S. Thorne, Membrane viewpoint on black holes: Properties and evolution of the stretched horizon, Phys. Rev. D 33 (1986) 915–941.

[4] S. A. Hayward, S. Mukohyama, M. C. Ashworth, Dynamic black-hole entropy, Phys. Lett. A 256 (1999) 347–350, arXiv:gr-qc/9810006.

[5] K. Parattu, B. R. Majhi, T. Padmanabhan, Structure of the gravitational action and its relation with horizon thermodynamics and emergent gravity paradigm, Phys. Rev. D 87 (2013) 124011, arXiv:1303.1535.

[6] V. Faraoni, *Cosmological and Black Hole Apparent Horizons*, Lecture Notes in Physics 907, Springer, Cham, 2015, Sec. 3.15.

[7] W. G. Unruh, Notes on black-hole evaporation, Phys. Rev. D 14 (1976) 870–892.

[8] F. Hopfmüller, L. Freidel, Gravity degrees of freedom on a null surface, Phys. Rev. D 95 (2017) 104006, arXiv:1611.03096.

[9] F. Hopfmüller, L. Freidel, Null conservation laws for gravity, Phys. Rev. D 97 (2018) 124029, arXiv:1802.06135.

[10] G. Odak, A. Rignon-Bret, S. Speziale, General gravitational charges on null hypersurfaces, JHEP 12 (2023) 038, arXiv:2309.03854.

[11] S. Carlip, C. Teitelboim, The off-shell black hole, Class. Quantum Grav. 12 (1995) 1699–1704, arXiv:gr-qc/9312002.

[12] E. Bianchi, W. Wieland, Horizon energy as the boost boundary term in general relativity and loop gravity, arXiv:1205.5325.

[13] E. Frodden, A. Ghosh, A. Perez, Quasilocal first law for black hole thermodynamics, Phys. Rev. D 87 (2013) 121503, arXiv:1110.4055.

[14] R.-G. Cai, S. P. Kim, First law of thermodynamics and Friedmann equations of FRW universe, JHEP 02 (2005) 050, arXiv:hep-th/0501055.

[15] C. W. Misner, D. H. Sharp, Relativistic equations for adiabatic, spherically symmetric gravitational collapse, Phys. Rev. 136 (1964) B571–B576.

[16] S. A. Hayward, Unified first law of black-hole dynamics and relativistic thermodynamics, Class. Quantum Grav. 15 (1998) 3147–3162, arXiv:gr-qc/9710089.

[17] V. Faraoni, Cosmological apparent and trapping horizons, Phys. Rev. D 84 (2011) 024003, arXiv:1106.4427.

[18] S.-B. Kong, Effective pressure of the FRW universe, Ann. Phys. 481 (2025) 170163, doi:10.1016/j.aop.2025.170163.

[19] S. W. Hawking, Particle creation by black holes, Commun. Math. Phys. 43 (1975) 199–220.

[20] G. W. Gibbons, S. W. Hawking, Cosmological event horizons, thermodynamics, and particle creation, Phys. Rev. D 15 (1977) 2738–2751.

[21] S. A. Hayward, Gravitational energy in spherical symmetry, Phys. Rev. D 53 (1996) 1938–1949, arXiv:gr-qc/9408002.