Horizon Boundary Mechanics:
Force, Pressure, and Work from the Einstein Boundary Potential

J. T. Tyler[1,*]

[1] *La Jolla, CA, USA*

(Dated: Boundary-mechanics working revision: 18 September 2026)

Horizon boundary mechanics resolves the Einstein horizon boundary potential into a surface-stress response to area and a force–displacement response to acceleration length. For a prescribed normal flow and clock, the potential's complete differential separates surface-stress work from work conjugate to the signed acceleration length set by surface gravity. The force is proportional to horizon area and the square of surface gravity, and force per unit area defines the normal-scale pressure. The mechanical construction applies wherever this specified potential and normal-rate prescription apply. Its canonical realization and conversion to geometric displacement or volume work are determined by the boundary setting. In the selected nonexpanding Einstein null sector, the force is the mixed-action response to the length source; the stationary area–boost corner pair independently gives the same coefficient. No thermal input is needed for this derivation.

Rindler supplies an operational proper-distance normalization without an enclosed source mass. Stationary black-hole realizations include Schwarzschild, where the acceleration length is twice the horizon radius and the thermal–mechanical Smarr products coincide. Spherical radius work requires the relation between radius and acceleration length, including its changing-ratio contribution. As a major cosmological application, the separately calibrated Cai–Kim FRW apparent horizon gives a screen pressure equal to one third of the bulk energy density and a first law combining temperature–entropy and pressure–volume contributions, using the established Einstein bulk-energy relation. Screen pressure organizes the horizon curvature, cosmic acceleration, and conservation equations; the full Hayward rate retains its distinct pressure-contrast and work allocation. In standard slow-roll inflation, the established tensor spectrum and its scale dependence are expressed through the screen pressure and horizon entropy.

Hawking–Unruh thermality is introduced afterward and relates the same normal scale to temperature, entropy, and the boost charge. Part I presents the common mechanics and its principal realizations. Part II supplies canonical grounding, membrane and normal-force-balance relations, finite dynamical and stationary higher-curvature extensions, and the clock, sign, integrability, and moving-boundary limits. The framework describes boundary-potential mechanics, not a new material stress or a replacement for Einstein's equations.

HOW TO READ THIS PAPER

The organizing object is the Einstein boundary potential, not a particular black hole or cosmology. Part I states its area and acceleration-length responses and then supplies the Rindler, Schwarzschild, and Cai–Kim FRW dictionaries. Part II establishes the primary null-source response and its Hamilton–Jacobi meaning, checks it through the stationary area–boost pair, and examines the conditions for each physical realization. The boost product is an explanation of the same mechanics, not a third independent proof. FRW is a major cosmological application with a separately prescribed rate and bulk-energy identification; it is not the restriction defining the framework. Read the stationary and cosmological sections for applications, and the generality discussion and moving-screen appendix for the limits of canonical extension.

* jttyler101@gmail.com

PART I. BOUNDARY MECHANICS AND PRINCIPAL REALIZATIONS

Common mechanics and canonical scope. The common mechanics below follows from the displayed Einstein boundary potential with a prescribed physical flow, clock, and sign convention. Its normal form is not restricted to one horizon geometry; each realization supplies its rate and displacement dictionary. The canonical statement in Part II additionally specifies a mixed rate-source polarization, boundary and corner terms, and variations tangent to a non-expanding null sector. Varying the rate source means comparing nearby boundary problems; it is not a variation that keeps that source fixed. Rindler and Schwarzschild realize the stationary construction. Cai–Kim FRW is a separate constrained-screen realization with an instantaneous-equilibrium calibration. The thermal representation is introduced after the mechanical response.

Acceleration length and the parent potential

On the nonexpanding null sector, the Einstein mixed scalar source reduces to inaffinity, or surface gravity κ in acceleration units. The generic null scalar source also contains expansion, as made explicit in Part II. On either nonzero signed branch define

$$L_{\rm s} \equiv \frac{c^2}{\kappa}, \qquad L_\kappa \equiv |L_{\rm s}| = \frac{c^2}{|\kappa|}. \tag{S1}$$

Up to a dimensionless constant, this is the only length constructible from c and κ without another scale. Rindler geometry fixes the constant to one because c^2/a is the exact observer–horizon proper distance. A spherical radius realizes this normal scale only after a sector supplies a relation, such as $L_\kappa = R_A$ in Cai–Kim FRW or $L_\kappa = 2R_H$ in Schwarzschild.

The established scalar area–normal-rate boundary potential is ordinarily written in the familiar surface-gravity form. Equation (S1) permits the equivalent acceleration-length form:

$$\boxed{U_\partial(A,\kappa) = \frac{Ac^2\kappa}{8\pi G}, \qquad U_\partial(A, L_{\rm s}) = \frac{Ac^4}{8\pi G L_{\rm s}}.} \tag{S2}$$

Here $A = \int_S \epsilon_S$ is intrinsic cut area, not a radius-based definition. For a Rindler rectangle $A = \Delta y\, \Delta z$; only a round sphere supplies $A = 4\pi R^2$. Transverse patch size and the selected normal acceleration length remain independent unless a physical family relates them.

Conventional mass first laws prescribe the normal evolution and vary a different Hamiltonian energy. Here the boundary potential itself is varied in both of its displayed coordinates, so its scale leg is unavoidable.

Boost-charge interpretation of the same mechanics

The normal-rate potential can be read before any thermal identification. Let J_η denote the Einstein charge conjugate to a dimensionless normal boost angle, and let ω_s be its signed rate:

$$J_\eta = \frac{Ac^3}{8\pi G}, \qquad \omega_s = \frac{\kappa}{c} = \frac{c}{L_{\rm s}}, \qquad \boxed{U_\partial = \omega_s J_\eta, \quad U_\partial L_{\rm s} = cJ_\eta.} \tag{S3}$$

The charge has units of action, not ordinary rotational angular momentum. Multiplication by the normal boost rate gives the energy-valued boundary potential. At fixed area its energy–length product is independent of the rate: $U_\partial L_{\rm s}/c = J_\eta$. With positive magnitudes the same relation is $|U_\partial| L_\kappa = cJ_\eta$. This is an explanatory consolidation of the established area–boost structure, not a new independent conserved energy. Its stationary canonical derivation is given in Part II.

The two contributions of the complete differential are

$$\boxed{dU_\partial = \omega_s\, dJ_\eta + J_\eta\, d\omega_s = \Pi_\partial\, dA - F_L\, dL_{\rm s}.} \tag{S4}$$

The first changes the area charge at a prescribed rate; the second changes the rate at prescribed charge. On the positive branch, only after Hawking–Unruh thermality is supplied, these become $T\,dS$ and $S\,dT$, respectively. Mechanical rate and charge should not be called thermal variables before that additional identification.

The minimal mechanical differential

Taking the complete differential of Eq. (S2) gives

$$\boxed{dU_\partial = \frac{U_\partial}{A}\,dA - \frac{U_\partial}{L_\mathrm{s}}\,dL_\mathrm{s} = \Pi_\partial\,dA - F_L\,dL_\mathrm{s}.} \tag{S5}$$

The two coefficients are

$$\Pi_\partial \equiv \frac{U_\partial}{A} = \frac{c^4}{8\pi G L_\mathrm{s}} = \frac{c^2\kappa}{8\pi G}, \tag{S6}$$

$$F_L \equiv \frac{U_\partial}{L_\mathrm{s}} = \frac{Ac^4}{8\pi G L_\mathrm{s}^2} = \frac{A\kappa^2}{8\pi G} = \frac{U_\partial}{c^2}\,\kappa. \tag{S7}$$

Equation (S6) is the established two-dimensional horizon surface pressure or tension, conjugate to area. With $M_\partial \equiv U_\partial/c^2$, Eq. (S7) has the exact mass-equivalent form

$$\boxed{F_L = M_\partial \kappa.} \tag{S8}$$

This is an identity for the boundary mass equivalent, not a separate equation of motion for a material body. Dividing by transverse area gives the normal-scale force-per-area pressure

$$\boxed{P_L \equiv \frac{F_L}{A} = \frac{U_\partial}{AL_\mathrm{s}} = \frac{c^4}{8\pi G L_\mathrm{s}^2} = \frac{\kappa^2}{8\pi G}.} \tag{S9}$$

The primary work term is $F_L\,dL_\mathrm{s}$. Pressure is its coefficient per transverse area:

$$\boxed{dU_\partial = \Pi_\partial\,dA - F_L\,dL_\mathrm{s}, \qquad F_L\,dL_\mathrm{s} = P_L A\,dL_\mathrm{s}.} \tag{S10}$$

No volume definition is needed. When useful, introduce the volume-valued one-form $\vartheta_V \equiv A\,dL_\mathrm{s}$. On the independent (A, L_s) source space it is generally inexact:

$$d\vartheta_V = dA \wedge dL_\mathrm{s}, \qquad d(AL_\mathrm{s}) = A\,dL_\mathrm{s} + L_\mathrm{s}\,dA. \tag{S11}$$

At fixed area $\vartheta_V = d(AL_\mathrm{s})$; on a specified spherical family it may instead equal the differential of a geometric volume. The name pressure refers to force per area conjugate to the stated scale, and does not assert a universal proper displacement or enclosed volume.

The established surface stress is mechanically resolved by

$$\boxed{\Pi_\partial = P_L L_\mathrm{s}, \qquad |\Pi_\partial| = P_L L_\kappa, \qquad P_L = -\frac{\partial |\Pi_\partial|}{\partial L_\kappa}.} \tag{S12}$$

The compact Euler chain is

$$\boxed{U_\partial = \Pi_\partial A = F_L L_\mathrm{s} = P_L A L_\mathrm{s}.} \tag{S13}$$

The known surface tension already packaged boundary energy per area; the acceleration-length factorization exposes its normal force, displacement, and pressure-work content.

Thermal translation and complementary scale laws

Where Hawking–Unruh/KMS thermality applies, it independently selects the same positive scale through $c\Delta\tau_E = 2\pi L_\kappa$. On the conventional positive thermodynamic branch, $U_\partial = TS$ and $\Pi_\partial\,dA = T\,dS$, so the same differential becomes

$$\boxed{dU_\partial = T\,dS - F_L\,dL_\kappa = T\,dS - P_L\,\vartheta_V.} \tag{S14}$$

Surface gravity sets two complementary laws of the same positive scale:

$$\boxed{k_B T = \frac{\hbar|\kappa|}{2\pi c} = \frac{\hbar c}{2\pi L_\kappa}, \qquad P_L = \frac{\kappa^2}{8\pi G} = \frac{c^4}{8\pi G L_\kappa^2}.} \tag{S15}$$

Hawking temperature is the semiclassical thermal law; P_L is its classical mechanical counterpart, conjugate per unit area to a change of acceleration length. Combining the two normalizations gives

$$\boxed{\frac{k_B T}{P_L L_\kappa} = 4\ell_{\mathrm{P}}^2, \qquad \frac{S}{A} = \frac{P_L L_\kappa}{T} = \frac{k_B}{4\ell_{\mathrm{P}}^2}.} \tag{S16}$$

Pressure times acceleration length supplies the classical boundary energy per area; temperature converts it into the displayed entropy density. If entropy is reconstructed from $T\,dS = \Pi_\partial\, dA$, the integration gives $S = k_B A/(4\ell_{\mathrm{P}}^2) + S_0$; $S_0 = 0$ is the standard Einstein/Wald normalization used throughout.

Stationary and cosmological realizations

Cai–Kim FRW. For an FRW apparent horizon in the instantaneous Cai–Kim prescription,

$$L_\kappa = R_A, \qquad U_\partial = E_A, \qquad P_L = \frac{\rho}{3}, \qquad \boxed{dE_A = T\,dS_A - P_L\,dV_A.} \tag{S17}$$

Here acceleration length is the areal screen radius and $\vartheta_V = A_A\,dR_A = dV_A$, where $V_A = 4\pi R_A^3/3$ is the areal volume. For nonzero spatial curvature it is not the proper volume of the cosmic-time spatial ball. With the Einstein coupling $\kappa_{\mathrm{E}} \equiv 8\pi G/c^4$, the Friedmann constraint and acceleration equation take the compact pressure form

$$\boxed{\frac{1}{R_A^2} = \kappa_{\mathrm{E}} P_L, \qquad \frac{2\ddot{a}}{ac^2} = -\kappa_{\mathrm{E}}(P_L + p_{\mathrm{m}}).} \tag{S18}$$

Here P_L fixes the transverse horizon-curvature scale, while $P_L + p_{\mathrm{m}}$ fixes the normal expansion curvature. Full Kodama–Hayward cross-focusing instead depends on the pressure contrast,

$$\boxed{\kappa_H^{\mathrm{s}} = -\frac{c^2 R_A}{4}\kappa_{\mathrm{E}}(P_L - p_{\mathrm{m}}), \qquad P_L^H = \frac{(P_L - p_{\mathrm{m}})^2}{16 P_L}.} \tag{S19}$$

Equivalently, Faraoni's established trace dependence can be packaged as

$$\Sigma_{\mathrm{tr}} \equiv \frac{(\rho - 3p_{\mathrm{m}})V_A}{c^2 A_A} = \frac{P_L - p_{\mathrm{m}}}{\kappa_{\mathrm{CK}}}, \qquad \boxed{\kappa_H^{\mathrm{s}} = -2\pi G \Sigma_{\mathrm{tr}}.} \tag{S20}$$

Here Σ_{tr} is the signed bulk stress–energy trace, converted to a mass equivalent and distributed over the horizon area. It is not assumed to be a membrane energy density or a literal shell source. The detailed status and operational tests are treated in Sec. VI.

Schwarzschild. For a nonrotating black hole with Killing time normalized at infinity,

$$L_\kappa = 2R_H, \qquad U_\partial = TS = F_L L_\kappa = \frac{Mc^2}{2}, \qquad F_L = \frac{c^4}{8G}. \tag{S21}$$

The Smarr relation becomes

$$\boxed{Mc^2 = TS + F_L L_\kappa, \qquad TS = F_L L_\kappa = \frac{Mc^2}{2}.} \tag{S22}$$

This is an exact Euler–Smarr balance. Both products equal the parent potential; their gravitational scaling motivates a stationary virial interpretation, while a kinetic/potential identification requires an additional derivation. The scale work also has the exact radial form

$$F_L\,dL_\kappa = (M\kappa)\,dR_H = 2P_L\,dV_H, \qquad 2F_L = M\kappa = \frac{c^4}{4G}. \tag{S23}$$

Here $V_H = 4\pi R_H^3/3$ and the area leg of dU_∂ is retained. Along the Schwarzschild family,

$$d(TS) = d(F_L L_\kappa) = \frac{1}{2}\, d(Mc^2), \qquad Mc^2 - TS = F_L L_\kappa. \tag{S24}$$

The first equality follows from the common parent potential; the second identifies the scale product with the formal on-shell Helmholtz free energy. The different differential channels are $\Pi_\partial\, dA$ and $-F_L\, dL_\kappa$, not independently variable thermal and mechanical energies.

Rindler. For a uniformly accelerated observer in flat spacetime,

$$\kappa = a, \qquad L_\kappa = \frac{c^2}{a}, \qquad P_L = \frac{a^2}{8\pi G}, \qquad U_\partial = \frac{Aac^2}{8\pi G}. \tag{S25}$$

The acceleration length is literally the observer–horizon proper distance, the curvature radius of the accelerated worldline, and the Euclidean thermal-orbit radius. No source mass, center, curvature, or spherical radius is required.

Spherical energy–length interpretation

On a spherical marginal horizon, the geometric Misner–Sharp value is $E_{\rm MS} = c^4 R/(2G)$. Substituting $A = 4\pi R^2$ into the same potential gives

$$\boxed{U_\partial = E_{\rm MS}\frac{R}{L_{\rm s}}, \qquad U_\partial L_{\rm s} = cJ_\eta = E_{\rm MS} R.} \tag{S26}$$

For $\kappa > 0$, the energy ratio is $U_\partial/E_{\rm MS} = R/L_\kappa$: it equals one for Cai–Kim FRW and one half for infinity-normalized Schwarzschild. This spherical interpretation of the area–boost product is not a new independent charge law. $E_{\rm MS} \propto R$ here, so it must vary with R in a differential. No enclosed Misner–Sharp energy or spherical projection is required for the general Rindler patch potential.

Radius work and the Smarr scaling identity

For a spherical cut, introduce

$$A = 4\pi R^2, \qquad M_R \equiv \frac{c^2 R}{2G}, \qquad \alpha_R \equiv \frac{L_{\rm s}}{R} = \frac{c^2}{\kappa R}, \qquad F_R \equiv \frac{U_\partial}{R} = M_R \kappa. \tag{S27}$$

Here M_R equals the Misner–Sharp mass at a spherical marginal horizon; it is not a universal total mass for arbitrary screens. On either nonzero signed branch the exact change of variables gives

$$\boxed{F_L\, dL_{\rm s} = F_R\, dR + U_\partial \frac{d\alpha_R}{\alpha_R}, \qquad dU_\partial = F_R\, dR - U_\partial \frac{d\alpha_R}{\alpha_R}.} \tag{S28}$$

The area leg is $\Pi_\partial\, dA = 2F_R\, dR$. The product identities below hold throughout this source chart. At fixed α_R, $F_R = c^4/(2G\alpha_R)$ is constant and the potential is linear in R:

$$\boxed{U_\partial = F_R R = 3P_R V, \qquad P_R \equiv F_R/A = \alpha_R P_L, \qquad R\left(\frac{\partial U_\partial}{\partial R}\right)_{\alpha_R} = U_\partial.} \tag{S29}$$

The areal volume is $V = 4\pi R^3/3$. Cai–Kim FRW has $\alpha_R = 1$, $M_R = M_A$, and $U_\partial = E_A$; Schwarzschild has $\alpha_R = 2$, $M_R = M$, and $Mc^2 = 2U_\partial = 2F_R R$. These are constrained scaling relations, not additional independent terms in the mass first law. The radius form makes Smarr homogeneity explicit. The same homogeneity is already present in the acceleration-length potential as $2A\,\partial_A U_\partial + L_{\rm s}\,\partial_{L_{\rm s}} U_\partial = U_\partial$. Sections II D and V G give the canonical normalization and the limits of the mass–radius interpretation, including rotating and charged examples.

The complete radius formulation is as general as the acceleration-length formulation within this spherical scalar sector. Only its truncation to $F_R dR$ requires a constant scale ratio. Their common foundation is the area–normal-rate pair: A measures transverse size, whereas L_s represents the selected normal rate.

A useful boundary-data construction makes the connection to Rindler explicit. On its positive branch impose an equivalent sphere $A_{\mathrm{eq}} = 4\pi L_\kappa^2$ and $V_{\mathrm{eq}} = 4\pi L_\kappa^3/3$. Then

$$U_{\mathrm{eq}} = \frac{c^4 L_\kappa}{2G}, \qquad \frac{U_{\mathrm{eq}}}{V_{\mathrm{eq}}} = 3P_L, \qquad \frac{2G(U_{\mathrm{eq}}/c^2)}{c^2} = L_\kappa. \tag{S30}$$

These are exactly the Cai–Kim spherical boundary formulas. The completion fixes the otherwise independent transverse area; it does not give the planar Rindler horizon an intrinsic spherical radius or Minkowski spacetime a Misner–Sharp mass. De Sitter realizes these spherical data geometrically. Section V B explains both the exact match and the distinction between horizon radius, normal proper distance, and clock-normalized acceleration length.

Boundary inverse-energy or Compton-equivalent consequence

Restrict this subsection to the positive-κ branch. Any specified positive energy scale E defines the quantum inverse-energy length $\hbar c/E$. Applying that construction to the pressure-carrying boundary potential gives

$$\bar{\lambda}_\partial \equiv \frac{\hbar c}{U_\partial}. \tag{S31}$$

It has the mathematical form of a reduced Compton wavelength, but U_∂/c^2 is generally a boundary mass equivalent rather than particle rest mass. The Euclidean thermal period in light-travel units is $C_E \equiv c\Delta\tau_E = 2\pi L_\kappa$. Since $U_\partial = TS$,

$$\boxed{\frac{S}{k_B} = \frac{C_E}{\bar{\lambda}_\partial} = \frac{2\pi L_\kappa}{\bar{\lambda}_\partial}, \qquad \frac{\bar{\lambda}_\partial}{L_\kappa} = \frac{8\pi \ell_{\mathrm{P}}^2}{A}.} \tag{S32}$$

Horizon entropy therefore measures the separation between the macroscopic thermal scale of the selected normal flow and the quantum inverse-energy scale of its boundary potential. This is an exact macroscopic relation within the Einstein–Hawking framework, not a claim that the horizon is a particle or is composed of literal wavelength-sized cells. Equivalently,

$$S = \frac{2\pi k_B U_\partial L_\kappa}{\hbar c}, \tag{S33}$$

which reproduces the familiar energy–radius product in Schwarzschild and Cai–Kim FRW and suggests $U_\partial L_\kappa$ as the more general normal-flow boundary product.

PART II. CANONICAL FOUNDATIONS, SCOPE, AND EXTENSIONS

Purpose of Part II. The compact derivation in Part I is self-contained once its inherited horizon-mechanics inputs are accepted. The task here is canonical certification and scope control: to derive the source and response from the covariant Einstein boundary theory, recover the same force through independent structures, and determine exactly where the compact normal form applies, where it must be qualified, and which additional consequences survive.

What this part establishes. The equations above are algebraically complete once their stated horizon-mechanics inputs are accepted. Part II supplies the covariant proof certificate connecting them to general relativity. It derives the permitted L_{s} source variation from the mixed null symplectic potential (Sec. II), independently recovers F_L from the area–boost corner action, and establishes the inherited clock and gauge meaning, signed branches and $\kappa = 0$ limit, membrane pullback and normal force balance, Cai–Kim/Hayward distinction, off-family integrability limit, finite dynamical support, and higher-curvature extensions. At $\kappa = 0$ the reciprocal chart moves to infinity and the regular parent description must be written in κ.

FROM ONE ACTION TO SEVERAL BOUNDARY DESCRIPTIONS

Einstein gravity on a finite region has a bulk action, boundary terms, and corner terms. Its variation separates field equations from boundary responses, schematically

$$\delta I = \int_M E \cdot \delta\Phi + \Theta_{\partial M} + \delta I_{\mathrm{boundary}} + \delta I_{\mathrm{corner}}.$$

Here Φ denotes metric and matter fields, $E = 0$ denotes their equations, and Θ is a field-space one-form, not an energy function. Evaluating on solutions gives a Hamilton–Jacobi functional of the admitted boundary data. Changing its polarization exchanges which member of a canonical pair is prescribed. Changing its clock or reference terms can change the physical response, even when the bulk equations are unchanged.

The null scalar potential and stationary area–boost corner give two routes to the same coefficient in their common sector. The surface Hamiltonian supplies the area/rate differential interpretation. The Brown–York timelike stress, spherical Kodama–Noether potential, and matter-flux laws are related descriptions with different geometric or variational inputs; they are not automatically one identical Hamiltonian. A complete Hamiltonian variation can contain symplectic, generator, transport, and flux corrections to the Noether surface term [11, 12, 21]. The analysis below specifies these distinctions rather than introducing several unrelated force laws.

I. INTRODUCTION

Part I displayed the compact result in its minimal A–L_{s} normal form. This part begins from the same physical setting and asks what remains to be certified by the covariant theory.

Horizon mechanics naturally singles out area and surface gravity. Horizon thermodynamics repackages the same pair as entropy and temperature. What is less often made explicit is the mechanical content of varying the normal rate itself. In a stationary first law the horizon generator and its clock normalization are normally held fixed, so the area variation is visible while the complementary $A\,\delta\kappa$ direction is absent or absorbed by the definition of the charge. This is appropriate for that ensemble, but it leaves open a different boundary-value question: if the prescribed normal rate is allowed to vary as a source, what is its response?

The selected clock, generator normalization, relational cut, and equilibrium or non-expanding conditions are not special restrictions invented for P_L. They are inherited from the same horizon-mechanics sector in which the standard first law, membrane surface stress, and Hawking temperature define the same κ. The added construction is to open the reciprocal normal-rate source direction and take the full differential of the Einstein boundary potential $U_\partial(A, L_{\mathrm{s}})$, rather than only the Hamiltonian mass variation for a prescribed evolution. The identification $U_\partial = TS$ belongs to the later thermal translation.

The reciprocal length

$$L_{\mathrm{s}} = \frac{c^2}{\kappa} \tag{1}$$

turns that question into a mechanical one. Surface gravity is the regular geometric rate; acceleration length is its reciprocal mechanical scale. On each nonzero branch they contain the same information, but κ remains the regular parent coordinate at a zero crossing while L_{s} moves to infinity. The latter is the light-travel length associated with the inverse surface-gravity timescale. In Rindler spacetime it is exactly the proper distance from a uniformly accelerated observer to that observer's horizon. For a stationary black hole it is an inverse redshift-normalized surface-gravity scale. In spherical dynamics its Hayward analogue is an inverse normal cross-focusing scale.

On stationary horizons admitting a regular Hawking–Unruh/KMS state, the thermal meaning of this reciprocal scale is exact rather than a dimensional resemblance. Analytic continuation of the normal time turns Lorentzian boost flow into rotation in the Euclidean normal two-plane. The horizon cut is the fixed-point set of that rotation, smoothness fixes a 2π angular period, and $c\Delta\tau_E = 2\pi L_\kappa$. Equivalently, the first nonzero bosonic Matsubara frequency is $\omega_1^{\mathrm{E}} = |\kappa|/c$, so $L_\kappa = c/\omega_1^{\mathrm{E}}$. The Euclidean/KMS periodicity and Matsubara spectrum are established results; the structural use made here is to identify that same reduced thermal length as the mechanical boundary source whose Einstein-action response is pressure [7, 33–36].

The common Einstein boundary potential used below is

$$U_\partial(A, L_{\mathrm{s}}) = \mathcal{T}_G \frac{A}{L_{\mathrm{s}}}, \qquad \mathcal{T}_G \equiv \frac{c^4}{8\pi G}, \tag{2}$$

whose fixed-area derivative gives

$$F_L = -\left(\frac{\partial U_\partial}{\partial L_{\mathrm{s}}}\right)_A, \qquad P_L \equiv \frac{F_L}{A} = \frac{c^4}{8\pi G L_{\mathrm{s}}^2} = \frac{\kappa^2}{8\pi G}. \tag{3}$$

At the reduced level, Eqs. (1)–(3) form a self-contained conditional theorem: once the established parent potential and acceleration-length coordinate are accepted in the inherited horizon-mechanics sector, the force, pressure, work form, and principal sectoral corollaries follow. The purpose of the full analysis is not to repair that algebra but to provide canonical certification and scope control: derive the allowed source variation from the covariant gravitational boundary theory, reproduce the force independently from the corner action, and determine the clock, gauge, sign, integrability, and dynamical limits.

The generic scalar source on a null boundary is not surface gravity alone but

$$\mu = \widehat{\kappa} + \frac{1}{2}\theta, \tag{4}$$

where $\widehat{\kappa} = \kappa/c^2$ is the inverse-length inaffinity and θ is the null expansion. The clean $\kappa \leftrightarrow L_{\mathrm{s}}$ source chart therefore applies canonically on a clock-anchored non-expanding sector, not on every moving horizon. Other sectors can support the same algebra only after an independently established prescription for their normal rate and parent $A\kappa$ potential has been supplied.

The paper develops four additional consequences of the source response. First, the signed and magnitude relations

$$\Pi_\partial = P_L L_{\mathrm{s}}, \qquad |\Pi_\partial| = P_L L_\kappa, \qquad \Pi_\partial \equiv \frac{c^2\kappa}{8\pi G}, \tag{5}$$

link the new normal pressure to the established two-dimensional membrane pressure. The link is stronger than a comparison of units: for a nonuniform signed source on a cut, $D_A\Pi_\partial = -P_L D_A L_{\mathrm{s}}$; equivalently, on a fixed-sign branch, $D_A|\Pi_\partial| = -P_L D_A L_\kappa$. This is the coordinate-free pullback of the source-space response one-form to the physical cut and turns the usual membrane pressure gradient into a spatial manifestation of the normal scale response.

Second, the same substitution converts the gravitational-screen Young–Laplace equation into a normal force balance, using the screen normal rate and matching clock specified in Sec. IV. With the orientation in which the surface tension is $\gamma = -\Pi_\partial$, the exact signed equation and its positive-κ form are

$$\Delta p - \Pi_\partial H = \Delta p - P_L L_{\mathrm{s}} H = 0, \qquad \Delta p - P_L L_\kappa H = 0 \quad (\kappa > 0). \tag{6}$$

Here H is the mean curvature and Δp is the appropriate outside-minus-inside normal pressure. Equation (6) is not a new Einstein equation; it is an acceleration-length rewriting of an established normal projection of Einstein's equation. It does, however, answer directly the question of whether P_L can enter a genuine force-balance law.

Third, L_κ is a mechanical–thermal bridge scale:

$$L_\kappa = \frac{c^2}{|\kappa|} = \frac{\hbar c}{2\pi k_B T}. \tag{7}$$

Classical boundary mechanics assigns it the inverse-square law $P_L(L_\kappa) = c^4/(8\pi G L_\kappa^2)$, while quantum field theory assigns it the inverse-linear law $k_B T(L_\kappa) = \hbar c/(2\pi L_\kappa)$. Their ratio fixes the Planck-area entropy density.

Fourth, Schwarzschild displays an exact pointwise thermal–mechanical Euler–Smarr balance that motivates a stationary virial interpretation:

$$Mc^2 = TS + F_L L_\kappa, \qquad TS = F_L L_\kappa = \frac{Mc^2}{2}. \tag{8}$$

The two products are equal representations of the same potential even on the extended source space; its area and scale differential legs are distinct. Along the Schwarzschild family the products co-vary exactly, $d(TS) = d(F_L L_\kappa) = \frac{1}{2}d(Mc^2)$: adding mass increases both halves in lockstep because both products equal the same parent potential. A kinetic/potential interpretation remains an analogy, not a derived decomposition into independent energies. The formal on-shell Helmholtz free energy is $Mc^2 - TS = F_L L_\kappa$. This supports, but does not by itself prove, a kinetic-like versus potential/free-energy interpretation.

The novelty is not the algebraic chain rule $\kappa \mapsto c^2/\kappa$, nor the underlying $A\kappa$ potential. It is the identification of acceleration length as an admissible and mechanically privileged mixed-boundary source coordinate; the associated action response and clock, gauge, and integrability conditions; the normal and tangential force-balance relations it organizes; and the consequences across Rindler, stationary black-hole, cosmological, and dynamical sectors.

The remainder of Part II establishes the source, gauge, canonical, mechanical, and thermal structure in Sections II–IV. Section V treats stationary realizations and the thermal–mechanical Euler identities. Section VI develops the FRW screen law and Hayward focusing response. Section VII treats integrability and finite dynamical balance. Section VIII gives the Wald-entropy extension and Section IX states the contribution and its limits. Appendix C records a separate timelike boundary comparison, restricted FRW surface charges, and the unresolved problem of matching the full moving-screen response to a horizon rate.

II. NON-EXPANDING NULL-BOUNDARY SOURCE-RESPONSE PROPOSITION

A. Established null scalar pair

Let $\mathcal{N}$ be a null hypersurface foliated by spacelike cuts S. Let ξ^a be a chosen null evolution vector tangent to $\mathcal{N}$, and let q_{AB} and $\epsilon_S = \sqrt{q}\, d^2x$ be the induced metric and area form on a cut. Define the inaffinity and expansion by

$$\nabla_\xi \xi^a = \widehat{\kappa}_\xi \xi^a, \qquad \theta_\xi = \frac{1}{\sqrt{q}}\mathcal{L}_\xi \sqrt{q}. \tag{9}$$

The cut is a spacelike two-dimensional cross-section, whereas $\mathcal{N}$ is its three-dimensional null history. A null normal is also tangent to $\mathcal{N}$. Inaffinity measures failure of the selected generator parameter to be affine: its tangent differentiates into a multiple of itself. A null generator has no proper time, so this is not the proper acceleration of a material observer. The expansion is the fractional change of the area element along that generator. In the length-parameter convention both $\widehat{\kappa}_\xi$ and θ_ξ have inverse-length units. The shear instead describes area-preserving shape distortion. The scalar, vector, and trace-free tensor labels classify cut geometry; they do not posit extra matter species.

For the canonical formulas in this subsection set $c = 1$, take the null parameter v to have units of length, and retain $8\pi G$ explicitly. Hopfmüller and Freidel derived the null symplectic potential without initially fixing the diffeomorphism gauge. In four dimensions its bulk part has the schematic spin decomposition [11]

$$\Theta^{\text{bulk}}_{\mathcal{N}} = \frac{1}{8\pi G}\int_{\mathcal{N}} \left[\frac{1}{2}\widetilde{\sigma}^{AB}\delta\widetilde{q}_{AB} - \overline{\eta}_A \delta\xi^A - \left(\widehat{\kappa}_\xi + \frac{1}{2}\theta_\xi\right)\delta \ln\sqrt{q}\right]\epsilon_{\mathcal{N}}. \tag{10}$$

The first two terms are the spin-two and spin-one sectors. The spin-zero momentum is

$$\boxed{\mu_\xi = \widehat{\kappa}_\xi + \frac{1}{2}\theta_\xi.} \tag{11}$$

Writing $\epsilon_{\mathcal{N}} = dv \wedge \epsilon_S$, the scalar area polarization is

$$\Theta^{(0)}_A = -\frac{1}{8\pi G}\int_{\mathcal{N}} \mu_\xi\, \delta\epsilon_S\, dv. \tag{12}$$

Adding the exact variation of

$$B_\mu = \frac{1}{8\pi G}\int_{\mathcal{N}} \mu_\xi \epsilon_S\, dv \tag{13}$$

changes the scalar polarization without changing the symplectic two-form:

$$\Theta_{\mu}^{(0)} = \Theta_{A}^{(0)} + \delta B_{\mu} = \frac{1}{8\pi G} \int_{\mathcal{N}} \epsilon_S \, \delta\mu_\xi \, dv. \tag{14}$$

The elementary operation is $-p\,\delta q + \delta(pq) = q\,\delta p$. The first form displays area as the source; the second displays its mixed rate. Because $\delta^2 B_\mu = 0$, this boundary Legendre transform leaves the scalar symplectic two-form unchanged. The superscript (0) labels the scalar sector, not perturbative order. The one-half in μ_ξ is the four-dimensional specialization of the null geometric decomposition, not a coefficient chosen to reproduce the pressure. Hopfmüller and Freidel interpret μ as a gravitational boundary pressure and argue that fixing it is a natural clock boundary condition for null translations [12]. This established pressure is linear in μ and is not the quadratic force-per-area response introduced below.

Odak, Rignon-Bret, and Speziale subsequently obtained the covariant conformal/York polarization for general null variations [13]. In their conventions the spin-zero source is $\theta + 2\widehat{\kappa} = 2\mu$. Their conservative mixed conditions fix that quantity together with appropriate spin-two, generator, and corner data. The scalar condition is Robin-type because it contains metric derivatives. The use of μ as a mixed-boundary source is established. For clarity, the displayed polarization is selected by the boundary Legendre term B_μ, with orientation, cut identifications, generator, and remaining boundary and corner data prescribed. Its variation is evaluated on an extended space of sources. Conservative boundary conditions impose $\delta\mu = 0$ within each fixed-source problem; the response below is obtained by comparing nearby such problems. An additional source-dependent boundary functional can shift the response. No invariance under arbitrary counterterm changes is claimed.

B. Acceleration-length source representation

Restrict to a phase-space sector consisting of non-expanding null boundaries and variations tangent to that sector:

$$\theta_\xi = 0, \qquad \delta\theta_\xi = 0. \tag{15}$$

Then $\mu_\xi = \widehat{\kappa}_\xi$. Restore physical units by defining the acceleration-valued surface gravity

$$\kappa_\xi \equiv c^2 \widehat{\kappa}_\xi. \tag{16}$$

Expressing the null evolution in a prescribed physical clock τ gives Eq. (14) in the form

$$\Theta_{\kappa}^{(0)} = \frac{c^2}{8\pi G} \int d\tau \int_S \epsilon_S \, \delta\kappa_\xi. \tag{17}$$

For uniform stationary data, $\Theta_\kappa^{(0)}/\Delta\tau = (Ac^2/8\pi G)\delta\kappa$ is exactly the normal-rate leg of dU_∂. Consequently a coordinate change of that *already identified* canonical source transforms the response without another dynamical assumption. The nontrivial input is the matching action, clock, polarization, and admitted source space, not a second test of the chain rule. The complete exact differential dU_∂ must still be distinguished from its selected, generally nonexact, rate leg. On either nonzero signed branch define

$$L_{\mathrm{s}} \equiv \frac{c^2}{\kappa_\xi}, \qquad \delta\kappa_\xi = -\frac{c^2}{L_{\mathrm{s}}^2} \delta L_{\mathrm{s}}. \tag{18}$$

Substitution yields

$$\boxed{\Theta_{L}^{(0)} = -\int d\tau \int_S \epsilon_S \, \frac{c^4}{8\pi G L_{\mathrm{s}}^2} \, \delta L_{\mathrm{s}}.} \tag{19}$$

The local response density is therefore

$$\boxed{P_L = \frac{c^4}{8\pi G L_{\mathrm{s}}^2} = \frac{\kappa_\xi^2}{8\pi G}.} \tag{20}$$

For a source uniform on a finite cut,

$$F_L \equiv \int_S P_L \, dA = A P_L, \qquad \Theta_{L}^{(0)} = -\int d\tau \, F_L \delta L_{\mathrm{s}}. \tag{21}$$

For uniform sources, define the volume-valued source one-form

$$\vartheta_V^{(\delta)} \equiv A\,\delta L_{\mathrm{s}}. \tag{22}$$

Then

$$\boxed{\Theta_L^{(0)} = -\int d\tau\, P_L \vartheta_V^{(\delta)}.} \tag{23}$$

This is simply $F_L\,\delta L_{\mathrm{s}} = P_L A\,\delta L_{\mathrm{s}}$. The notation $\vartheta_V^{(\delta)}$ denotes a one-form, not the variation of a global volume function on the independent source space. At fixed area it is exact. The canonical result identifies the normal-scale force per area; its conversion to a spatial traction additionally requires an operational displacement and clock prescription.

For the selected mixed action, orientation, cuts, counterterm prescription, and fixed remaining sources, the local-in-time Hamilton–Jacobi relation is

$$F_L(\tau) = -\frac{\delta I_{\mathrm{os}}}{\delta L_{\mathrm{s}}(\tau)}. \tag{24}$$

If the source is constant through a stationary interval $\Delta\tau$, then $-(\partial I_{\mathrm{os}}/\partial L_{\mathrm{s}})_{\Delta\tau} = \Delta\tau F_L$. Here canonical response means the coefficient paired with the chosen source in the Einstein symplectic potential, equivalently the functional derivative of the mixed on-shell action. It is not a claim that P_L is a covariant local bulk stress tensor.

Operationally, the force derivative compares nearby boundary-value problems, each with its own prescribed value of the rate source, while the remaining sources are controlled. It is not a variation that both fixes and changes the same source within one problem. The field-space variation preserves the stated nullness, cut identification, and nonexpanding conditions. Fixing the physical clock prescription does not force $\delta\kappa = 0$, because the metric connection may vary. No existence theorem for arbitrary independent boundary assignments is implied.

In ordinary Hamilton–Jacobi theory an endpoint coordinate derivative of the on-shell action is a momentum. Here $\delta I_{\mathrm{os}} = -\int d\tau F_L \delta L_{\mathrm{s}}$ makes the local functional derivative a force: action divided by time and length has force units. The constant-source derivative across an interval instead has momentum units. The full variational problem retains its other response and endpoint terms, so this local statement does not imply a globally separate scale-energy function.

The corresponding extended-source two-form is

$$\Omega_L^{(0)} = \delta\Theta_L^{(0)} = \int d\tau \int_S \delta L_{\mathrm{s}} \wedge \delta(\epsilon_S P_L). \tag{25}$$

The boundary Legendre transform from Eq. (12) to Eq. (14) is the genuine change of polarization. The subsequent map $\kappa \mapsto c^2/\kappa$ is an invertible source coordinate change inside the rate-source polarization, not a second Legendre transform. For a uniform stationary interval,

$$\frac{\Omega^{(0)}}{\Delta\tau} = \delta A \wedge \delta\Pi_\partial = \delta L_{\mathrm{s}} \wedge \delta F_L. \tag{26}$$

These describe one scalar canonical pair in different polarizations; they are not two additional canonical degrees of freedom.

Any smooth invertible coordinate $q = f(\kappa)$ would transform the response by the chain rule. The action does not select L_{s} by algebra alone. Acceleration length is mechanically privileged because, up to a dimensionless factor, it is the unique length constructible from c and κ without another scale; Rindler proper distance fixes that factor to one. It is also the reduced inverse-temperature scale, the coordinate in which the response has pressure units, and the coordinate for which $F_L L_{\mathrm{s}} = U_\partial$. In Cai–Kim FRW it becomes the areal radius and converts exactly to spherical $P\,dV$ work.

The result may be summarized as follows.

> *Non-expanding null-boundary source-response proposition.* Let $\mathcal{N}$ be a four-dimensional non-expanding Einstein null boundary equipped with a prescribed physical evolution vector, fixed relational cuts, and variations tangent to the non-expanding sector. Assume that matter boundary sources and omitted symplectic flux are controlled separately. On either branch $\kappa_\xi \neq 0$, the signed acceleration length $L_{\mathrm{s}} = c^2/\kappa_\xi = 1/\widehat{\kappa}_\xi$ is an admissible reparameterization of the covariant mixed spin-zero source. Its Hamilton–Jacobi response density is $P_L = \kappa_\xi^2/(8\pi G)$, and a uniform finite cut carries $F_L = AP_L$.

The hypotheses are substantive. On a generic expanding null boundary the source is μ, so after defining $\mu_{\rm acc} = c^2\mu$ the clean reciprocal coordinate is

$$L_\mu = \frac{c^2}{\mu_{\rm acc}}, \tag{27}$$

not c^2/κ alone. At $\kappa = 0$ the reciprocal chart is singular, while the parent rate is regular. Indeed,

$$-P_L \delta L_{\rm s} = -\frac{\kappa_\xi^2}{8\pi G}\delta\left(\frac{c^2}{\kappa_\xi}\right) = \frac{c^2}{8\pi G}\delta\kappa_\xi. \tag{28}$$

$P_L \to 0$ at $\kappa \to 0$ does not make the parent variation vanish; it means that the L chart has reached infinity.

a. The scalar response when expansion is retained. There is a useful extension at the level of the same selected scalar one-form, not a complete moving-horizon theorem. With the clock measure fixed as above and $\mu_{\rm acc} = c^2\mu = \kappa + c^2\theta/2$, the scalar part is

$$\Theta_\mu^{(0)} = -\int d\tau \int_S \epsilon_S P_L\, \delta L_{\rm s} + \int d\tau \int_S \epsilon_S \frac{c^4}{16\pi G}\,\delta\theta. \tag{29}$$

This follows by substituting $\delta\mu = \delta\kappa/c^2 + \delta\theta/2$ into Eq. (14). The original coefficient survives as a partial scalar response; the independent expansion term remains. At fixed expansion it drops out, provided the remaining boundary and constraint conditions admit that variation. Alternatively, $L_\mu = 1/\mu$ absorbs the complete scalar source and gives $P_\mu = c^4/(8\pi G L_\mu^2)$, but L_μ is not generally $L_{\rm s}$. Neither representation removes vector, shear, matter, or corner terms. It therefore does not establish that a non-null FRW apparent-horizon worldtube has the same one-source response.

C. Independent area–boost derivation

The stationary corner sector supplies an independent check. Einstein gravity assigns the relative boost angle η of the boundary normals the corner momentum

$$J_\eta = \frac{c^3 A}{8\pi G} \tag{30}$$

The charge is the coefficient of a dimensionless relative Lorentz boost of the normal frames at the corner. It has action units and is not the black hole's rotational angular momentum; changing the area changes this charge. Calling it a charge therefore does not assert conservation during arbitrary horizon evolution [15–17]. For a uniform stationary interval of prescribed duration $\Delta\tau$,

$$\eta = \frac{\kappa_\xi \Delta\tau}{c} = \frac{c\Delta\tau}{L_{\rm s}}. \tag{31}$$

At fixed A and $\Delta\tau$,

$$J_\eta \delta\eta = -\Delta\tau F_L \delta L_{\rm s}, \qquad \Omega_{\rm corner} = \Delta\tau\, \delta L_{\rm s} \wedge \delta F_L. \tag{32}$$

The momentum conjugate to $L_{\rm s}$ in this stationary edge-mode sector is

$$p_L = J_\eta \frac{\partial\eta}{\partial L_{\rm s}} = -\frac{c\Delta\tau J_\eta}{L_{\rm s}^2} = -\Delta\tau \frac{A\kappa^2}{8\pi G} = -\Delta\tau F_L. \tag{33}$$

The quantity p_L has momentum units; F_L is the response per unit boundary time. It is distinct from the pressure P_L. A force appears because the boost angle is a rate integrated over time and $L_{\rm s}$ is the reciprocal of that rate.

Equations (17) and (32) provide two independent routes to the same coefficient. Therefore the central canonical algebra is not contingent on a thermodynamic analogy. Its remaining conditionality concerns the selected boundary ensemble, clock, cuts, and allowed counterterms, not the chain-rule calculation itself.

The compact dictionary of Part I follows from the same pair:

$$\omega_s = \frac{\kappa}{c}, \qquad U_\partial = \omega_s J_\eta, \qquad \boxed{U_\partial L_{\rm s} = c J_\eta = \frac{A c^4}{8\pi G}.} \tag{34}$$

The product $U_\partial L_{\rm s}/c$ is the boost charge itself. At fixed area, energy and reciprocal-rate length vary inversely. Under a constant physical-clock rescaling the product is unchanged, although U_∂, $L_{\rm s}$, and P_L separately change. The area and rate legs read $\omega_s dJ_\eta = \Pi_\partial dA$ and $J_\eta d\omega_s = -F_L dL_{\rm s}$. Thermality is unnecessary for these identities. Only with the subsequent Hawking–Unruh identification do they become TdS and SdT. For $\kappa < 0$ the thermal dictionary uses magnitudes; the mechanical identity above retains signed U_∂ and $L_{\rm s}$.

The integrated stationary relation $\eta = \kappa\Delta\tau/c$ uses the prescribed stationary normal flow. A generic moving cosmological screen has normal-connection and transport data not captured by its elementary velocity rapidity. The area–boost pair alone does not establish that integrated relation with an arbitrarily substituted dynamical surface gravity.

D. Canonical radius representation of the spherical scalar sector

Keep the clock, interval, cuts, orientation, and scalar-sector prescriptions of the preceding derivation. On uniform spherical cuts, (R, α_R) are coordinates on the same two-dimensional source space as $(A, L_{\rm s})$. Since $U_\partial = RF_R$ and $F_R = c^4/(2G\alpha_R)$, the rate-source potential becomes

$$\begin{aligned}\frac{\Theta^{(0)}}{\Delta\tau} &= \frac{Ac^2}{8\pi G}\,\delta\kappa = -F_R\,\delta R - U_\partial\frac{\delta\alpha_R}{\alpha_R}\\ &= R\,\delta F_R - F_R\,\delta R.\end{aligned} \tag{35}$$

Consequently,

$$\boxed{\frac{\Omega^{(0)}}{\Delta\tau} = 2\,\delta R\wedge\delta F_R, \qquad \frac{\Theta^{(0)}_A}{\Delta\tau} = -\Pi_\partial\,\delta A = -2F_R\,\delta R.} \tag{36}$$

Thus the full scalar pair is represented by R and $2F_R$, with momentum $-2\Delta\tau F_R$ in the displayed area polarization. The potentials differ by δU_∂ per unit prescribed time. Calling F_R alone the full radius momentum would miss a factor of two. It is the coefficient of the scale-work leg when $\delta\alpha_R = 0$; the area response is twice that coefficient. On a fixed-α_R one-parameter family, the pulled-back two-form vanishes, as it must. No extra canonical degree of freedom is created.

One can also define an area-equivalent $R = \sqrt{A/(4\pi)}$ for a finite nonspherical patch and make the same algebraic transformation. This does not supply a preferred center, a spherical volume, or a physical Misner–Sharp mass. Those interpretations require spherical geometry and, for M_R to be its actual enclosed mass, marginality.

The corner calculation gives the same result directly: $J_\eta\delta\eta/\Delta\tau = Ac^2\delta\kappa/(8\pi G)$ at fixed interval, so substituting $\kappa = c^2/(\alpha_R R)$ reproduces Eq. (35). This is a radius representation of the established area–rate and area–boost structures, not an independent canonical foundation. Its application to a generic expanding cosmological screen is not proved by this non-expanding-null argument; the Cai–Kim realization below remains a separate prescription.

III. CLOCK, GAUGE, AND THE STANDARD FIRST LAW

A. Auxiliary normal versus physical clock

Surface gravity is a rate relative to a normal evolution. To separate an auxiliary null-frame choice from a physical clock, write

$$\xi^a = f\ell^a, \qquad \widehat{\kappa}_\xi = f\widehat{\kappa}_\ell + \ell[f]. \tag{37}$$

Under the simultaneous change

$$\ell^a \longrightarrow e^\sigma \ell^a, \qquad f \longrightarrow e^{-\sigma} f, \tag{38}$$

the physical vector ξ^a is unchanged, and direct substitution shows that $\widehat{\kappa}_\xi$, κ_ξ, and $L_{\rm s}$ are unchanged. The construction is therefore independent of the auxiliary null representative for a fixed physical normal flow [14].

Changing the physical clock is different. If $\xi^a \to \alpha\xi^a$, then

$$\widehat{\kappa}_{\alpha\xi} = \alpha\widehat{\kappa}_\xi + \xi[\alpha], \qquad \kappa_{\alpha\xi} = \alpha\kappa_\xi + c^2\xi[\alpha]. \tag{39}$$

For constant α,

$$L_{\rm s} \longrightarrow \frac{L_{\rm s}}{\alpha}, \qquad P_L \longrightarrow \alpha^2 P_L. \tag{40}$$

The area and intrinsic two-geometry do not change. P_L is a well-defined response after a clock prescription has been fixed, but is not an intrinsic scalar of the cut alone. This dependence is inherited, not a new weakness of the pressure: the same is true of surface gravity, energy, temperature, membrane stress, and Hamiltonian charge, which are also defined relative to the selected evolution vector.

Rindler, Schwarzschild, and spherical dynamics use different clocks but perform the same geometric task. Rindler uses boost flow normalized by the accelerated observer's proper time; Schwarzschild uses Killing time normalized at infinity; spherical dynamics uses the Kodama flow or the Cai–Kim instantaneous thermal calibration. What is common is a prescribed evolution in the two-dimensional normal plane, not a universal observer-independent accelerometer reading.

For a proper-gauge direction X preserving the selected boundary conditions, the source and response descend individually when

$$X[L_{\rm s}] = 0, \qquad X[P_L \epsilon_S] = 0, \tag{41}$$

and the scalar two-form defines a separate reduced sector when it is basic,

$$\iota_X \Omega_L^{(0)} = 0, \qquad \mathcal{L}_X \Omega_L^{(0)} = 0. \tag{42}$$

These conditions hold in the stationary edge-mode sector when clock and cuts are fixed relationally or retained as edge data and only source-preserving degeneracies are quotiented. They fail if arbitrary physical-clock rescalings are declared gauge.

B. Why the standard first law has no additional work term

The usual stationary first law computes the Hamiltonian variation of a specified horizon evolution. On a fixed cut the covariant-phase-space expression has the schematic form

$$\delta H_\xi = \int_S \left(\delta Q_\xi - \xi \cdot \Theta - Q_{\delta\xi}\right), \tag{43}$$

with additional transport terms when the cut moves [21–23]. "Fixed generator" means $\delta\xi^a = 0$ in field space; it does not generally mean $\delta\kappa_\xi = 0$. The complete expression cancels or reallocates pieces caused only by changing the measuring generator, leaving the familiar area channel.

Hayward, Mukohyama, and Ashworth derived the spherical Kodama–Noether charge $\oint Q_K = A\widehat{\kappa}_H/(8\pi)$ in geometric units and explicitly questioned the usual suppression of $\delta\widehat{\kappa}$ [19]. Parattu, Majhi, and Padmanabhan showed that the two pieces of the Einstein surface-Hamiltonian variation become $T\delta S$ and $S\delta T$ in thermodynamically conjugate variables [20]. The acceleration-length representation gives the latter a direct mechanical coordinate on the positive thermodynamic branch:

$$S\,\delta T = -F_L \delta L_\kappa. \tag{44}$$

The absence of $S\,dT$ from the standard mass first law is therefore not a missing additive energy. It is the consequence of a different boundary-value problem: the usual law varies the Hamiltonian mass for a specified evolution, whereas Eq. (44) resolves the boundary potential $U_\partial = TS$ when the normal-rate source is varied.

IV. BOUNDARY EQUATION, PULLBACK, AND MECHANICAL–THERMAL SCALE

A. One potential and two pressures

For a stationary uniform cut, the chosen Einstein boundary functional, with no additional area-dependent integration term, gives

$$U_\partial(A, L_{\rm s}) = \mathcal{T}_G \frac{A}{L_{\rm s}}, \qquad \mathcal{T}_G = \frac{c^4}{8\pi G}. \tag{45}$$

Its first derivatives are

$$\Pi_\partial \equiv \left(\frac{\partial U_\partial}{\partial A}\right)_{L_\mathrm{s}} = \frac{c^4}{8\pi G L_\mathrm{s}} = \frac{c^2\kappa}{8\pi G}, \tag{46}$$

$$F_L \equiv -\left(\frac{\partial U_\partial}{\partial L_\mathrm{s}}\right)_A = \frac{c^4 A}{8\pi G L_\mathrm{s}^2}, \qquad P_L = \frac{F_L}{A}. \tag{47}$$

The quantities are dimensionally and variationally different. Π_∂ is energy per area or force per length, the pressure/tension of a two-dimensional fluid. P_L is force per transverse area, the normal scale pressure.

The pressure interpretation of Π_∂ is independently established in the Damour–Navier–Stokes equation and membrane paradigm, where the tangential momentum balance contains the pressure-gradient term $-D_A\Pi_\partial$ [24–27]. Depending on orientation and stress convention, the same magnitude is called pressure or tension. The pressure interpretation of P_L is instead fixed by Eq. (23). The relation between them is the Hessian identity

$$-\left(\frac{\partial \Pi_\partial}{\partial L_\mathrm{s}}\right)_A = \left(\frac{\partial F_L}{\partial A}\right)_{L_\mathrm{s}} = P_L = \frac{\Pi_\partial}{L_\mathrm{s}}. \tag{48}$$

It is useful to call Π_∂ the *surface pressure/tension* and P_L the *normal scale pressure.*

B. Displacement, force per area, and volume

The generalized force $F_L = -(\partial U_\partial/\partial L_\mathrm{s})_A$ and its pressure $P_L = F_L/A$ are defined before any volume is introduced. For a specified displacement coordinate x, holding area and the remaining sources fixed gives

$$F_L\, dL_\mathrm{s} = F_x^{(L)}\, dx, \qquad F_x^{(L)} = F_L \left(\frac{\partial L_\mathrm{s}}{\partial x}\right)_A. \tag{49}$$

If $dV = A\,dx$, the coefficient of that volume work is $P_x = P_L(\partial L_\mathrm{s}/\partial x)_A$. Thus P_L itself is the pressure associated with dV only when the chosen displacement has $dx = dL_\mathrm{s}$. A proper spatial displacement is not supplied by the units of L_s alone. Rindler provides an exact proper-distance realization; other sectors specify their own geometric dictionary.

On a physical family both area and scale usually vary. Its full pullback is

$$\frac{dU_\partial}{dx} = \Pi_\partial \frac{dA}{dx} - F_L \frac{dL_\mathrm{s}}{dx}. \tag{50}$$

The second term is the scale-work leg, not generally the negative total derivative. For the positive-branch product $\mathcal{V}_\partial = AL_\kappa$ one instead has

$$d\mathcal{V}_\partial = A\,dL_\kappa + L_\kappa\,dA, \qquad dU_\partial = 2\Pi_\partial\,dA - P_L\,d\mathcal{V}_\partial. \tag{51}$$

This explains why the Euler identity $U_\partial = P_L\mathcal{V}_\partial$ cannot be inserted into the original differential while leaving the area coefficient unchanged. The force–length form avoids that ambiguity entirely.

C. What the pullback statement means

The word pullback has a precise differential-geometric meaning. Fix a nonzero sign branch $\sigma = \mathrm{sgn}(\kappa)$, let S be a horizon cut, and regard the signed acceleration length as a map

$$L_\mathrm{s} : S \longrightarrow \mathbb{R}_\sigma, \qquad x \longmapsto L_\mathrm{s}(x), \qquad \mathbb{R}_\sigma \equiv \{L : \sigma L > 0\}. \tag{52}$$

On that one-dimensional source branch, define

$$\Pi(L) = \frac{c^4}{8\pi G L}, \qquad d\Pi = -P_L(L)\,dL, \qquad P_L(L) = \frac{c^4}{8\pi G L^2}. \tag{53}$$

The pullback $L_\mathrm{s}^*(d\Pi)$ is the one-form on S defined at a point x and tangent vector $X \in T_xS$ by

$$\big[L_\mathrm{s}^*(d\Pi)\big]_x(X) = d\Pi_{L_\mathrm{s}(x)}\big((dL_\mathrm{s})_x(X)\big). \tag{54}$$

In coordinates this is simply the chain rule:

$$\boxed{L_{\mathrm{s}}^*(d\Pi) = d(\Pi \circ L_{\mathrm{s}}) = -P_L\, dL_{\mathrm{s}}, \qquad D_A\Pi_\partial = -P_L D_A L_{\mathrm{s}}.} \tag{55}$$

On the same fixed-sign branch, $L_\kappa = |L_{\mathrm{s}}|$ and

$$\Pi_\partial = \sigma P_L L_\kappa, \qquad D_A\Pi_\partial = -\sigma P_L D_A L_\kappa, \qquad |\Pi_\partial| = P_L L_\kappa, \qquad D_A|\Pi_\partial| = -P_L D_A L_\kappa. \tag{56}$$

Consequently the pressure force in the tangential horizon-fluid equation can be written exactly as

$$-D_A\Pi_\partial = P_L D_A L_{\mathrm{s}} = \sigma P_L D_A L_\kappa. \tag{57}$$

The established membrane pressure-gradient one-form is therefore the pullback to the physical cut of the signed source-space response one-form $-P_L dL_{\mathrm{s}}$. Equivalently, the stress-magnitude gradient is the pullback of $-P_L dL_\kappa$.

This relation gives P_L an organizing role, but it does not make it ontologically more fundamental than every parent quantity. Π_∂ is the regular surface stress that appears directly in the tangential momentum equation, while P_L is its susceptibility with respect to the mechanically privileged reciprocal-rate coordinate. Moreover, on a generic expanding null boundary the still more general parent source is $\mu = \widehat{\kappa} + \theta/2$. The accurate hierarchy is therefore: μ is the generic null source; Π_∂ is the established linear surface stress on the selected non-expanding sector; and P_L is the normal pressure that generates the change of that stress with acceleration length.

Here D_A is the tangential covariant derivative on the cut (its index is not the total area A). It measures a spatial stress gradient, whereas δ compares boundary data in field space. "Pullback" means using $L_{\mathrm{s}}(x)$ to evaluate the source-space identity on the cut. This changes neither the membrane equation nor its other shear and momentum terms; it identifies which spatial pattern of the established surface stress is represented by an acceleration-length gradient.

D. A genuine normal force-balance equation

The same relation enters a normal, rather than tangential, projection of Einstein's equation. Freidel and Yokokura showed that a timelike gravitational screen obeys the non-equilibrium equations of a viscous bubble, including a generalized Young–Laplace equation [28]. Their screen tension is defined by the screen normal-rate variable γ_t. In their geometric-unit conventions set $\kappa_{\mathrm{scr}} = \gamma_t$ and define the corresponding $L_{\mathrm{s,scr}} = 1/\kappa_{\mathrm{scr}}$ and $P_{L,\mathrm{scr}} = \kappa_{\mathrm{scr}}^2/(8\pi G)$. Suppressing these screen subscripts only in this subsection, their orientation gives

$$\gamma = -\Pi_\partial = -P_L L_{\mathrm{s}}. \tag{58}$$

For a static screen the normal momentum balance is

$$0 = \Delta p + \gamma H = \Delta p - \Pi_\partial H, \tag{59}$$

where H is the mean curvature of the cut in the spatial screen and $\Delta p = p_{\mathrm{out}} - p_{\mathrm{in}}$ includes both matter and the appropriate intrinsic three-curvature contribution. Substitution of Eq. (58) gives the exact signed form

$$\boxed{\Delta p - P_L L_{\mathrm{s}} H = 0.} \tag{60}$$

In the positive-κ orientation this is $\Delta p - P_L L_\kappa H = 0$; reversing the screen orientation reverses the signed stress and pressure-jump convention together. P_L enters a genuine normal force balance: its pressure units are converted into curvature pressure by the dimensionless factor $L_{\mathrm{s}} H$, or by $L_\kappa H$ in positive magnitudes. In general the traction is $\Delta p = P_L L_{\mathrm{s}} H$, not P_L alone. Identifying this screen rate with a particular horizon surface gravity requires a matching flow, lapse, and normalization; it is not automatic.

The full dynamical screen equation, in the geometric units and notation of Ref. [28], is

$$-(D_{\boldsymbol{t}} + \theta_{\boldsymbol{t}})\theta_{\boldsymbol{n}} = 8\pi G\big[\Delta p - P_L L_{\mathrm{s}} H\big] - \frac{1}{2}\widetilde{\Theta}_{\boldsymbol{t}} : \Theta_{\boldsymbol{n}} - \varrho\,\omega^2 - \Delta\varrho. \tag{61}$$

Here the additional terms encode shear/viscous, normal-connection, and lapse-gradient effects. In a spherically symmetric, irrotational, constant-lapse reduction,

$$-\left(D_{\boldsymbol{t}} + \frac{3}{4}\theta_{\boldsymbol{t}}\right)\frac{\theta_{\boldsymbol{n}}}{8\pi G} = \Delta p - P_L L_{\mathrm{s}} H. \tag{62}$$

Equations (60)–(62) give the normal force-balance extension under this screen-rate dictionary. They are exact acceleration-length rewritings of an established gravitational-screen equation, not an additional field equation. They also clarify why no three-dimensional Navier–Stokes equation for P_L is required. The Damour equation governs tangential two-dimensional fluid momentum; the Young–Laplace/normal constraint governs normal screen motion. P_L links the two through $\Pi_\partial = P_L L_{\mathrm{s}}$, or through $|\Pi_\partial| = P_L L_\kappa$ in positive magnitudes.

E. Mechanical–thermal bridge and entropy density

For the thermodynamic magnitudes in this subsection, choose the positive-κ orientation. Then $L_{\mathrm{s}} = L_\kappa$, $\Pi_\partial = |\Pi_\partial|$, and $U_\partial = |U_\partial| = TS$. Reversing orientation leaves T, P_L, L_κ, and all magnitudes unchanged while reversing the signed stress and work form, as summarized in Appendix A. Einstein entropy and the Hawking–Unruh temperature are

$$S = \frac{k_B c^3 A}{4G\hbar} = \frac{k_B A}{4\ell_{\mathrm{P}}^2}, \qquad \ell_{\mathrm{P}}^2 \equiv \frac{G\hbar}{c^3}, \tag{63}$$

$$T = \frac{\hbar|\kappa|}{2\pi k_B c} = \frac{\hbar c}{2\pi k_B L_\kappa}, \qquad L_\kappa = \frac{c^2}{|\kappa|}. \tag{64}$$

Equation (64) identifies L_κ as a *mechanical–thermal bridge scale.* Its Euclidean meaning is strongest for a nonextremal stationary horizon admitting a regular Hawking–Unruh/KMS state. Wick rotation is the analytic continuation of the selected Lorentzian time,

$$t = -i\tau_E, \tag{65}$$

not a physical evolution of the spacetime. In the two-plane normal to a horizon cut, the local Lorentzian and Euclidean metrics take the Rindler forms

$$ds^2_{\perp,L} \simeq -\left(\frac{|\kappa|\rho}{c^2}\right)^2 c^2 dt^2 + d\rho^2, \qquad ds^2_{\perp,E} = d\rho^2 + \rho^2 d\vartheta^2, \qquad \vartheta = \frac{|\kappa|\tau_E}{c}. \tag{66}$$

The Lorentzian horizon is at $\rho = 0$. In the Euclidean normal two-plane this is the rotation fixed point; in the full spacetime the corresponding fixed-point set is the codimension-two horizon cut, not a single spacetime point. Smoothness at $\rho = 0$ requires $\vartheta \sim \vartheta + 2\pi$. Any other period produces a conical defect at the horizon. Therefore

$$\Delta\tau_E = \frac{2\pi c}{|\kappa|}, \qquad c\Delta\tau_E = 2\pi L_\kappa. \tag{67}$$

The reason this periodicity is thermal is standard quantum statistical mechanics. The canonical partition function $Z(\beta) = \mathrm{Tr}\, e^{-\beta H}$ is represented by evolution through imaginary time $\hbar\beta$ with the endpoints identified. Bosonic correlators are periodic and fermionic correlators antiperiodic around that imaginary-time circle; the Lorentzian KMS condition is the operator-algebraic equilibrium statement of the same analyticity and periodicity. For the Minkowski vacuum restricted to a Rindler wedge, the Bisognano–Wichmann result gives the exact KMS property with respect to boosts, while an accelerated detector obeys the corresponding Unruh detailed balance. For a regular stationary black-hole Euclidean saddle, the same smoothness condition fixes the Hawking temperature [7, 33–35].

Writing $\beta = (k_B T)^{-1}$, the bosonic Matsubara frequencies are

$$\omega_n^{\mathrm{E}} = \frac{2\pi n}{\hbar\beta}. \tag{68}$$

Equations (64) and (67) give

$$\Omega_\kappa \equiv \frac{|\kappa|}{c} = \frac{2\pi k_B T}{\hbar} = \omega_1^{\mathrm{E}}, \qquad L_\kappa = \frac{c}{\Omega_\kappa} = \frac{c}{\omega_1^{\mathrm{E}}}. \tag{69}$$

The standard content is $\hbar\beta = 2\pi c/|\kappa|$ together with the Matsubara spectrum. Equation (69) is an exact corollary, familiar in natural units as $1/|\kappa| = \beta/(2\pi)$, but it is not usually foregrounded as a horizon-mechanical coordinate. The new structural use here is that the same reduced Euclidean thermal length is also the source coordinate whose Einstein boundary response is P_L. It should not be confused with the Lorentzian wavelength of a typical emitted

Hawking quantum. In Rindler spacetime L_κ is literally the radius of the Euclidean orbit of the selected accelerated observer; for a globally normalized black-hole clock it is the Euclidean time period in light-travel units divided by 2π, not generally a proper radial distance [36].

The two laws carried by the same scale are

$$\boxed{P_L(L_\kappa) = \frac{c^4}{8\pi G L_\kappa^2}, \qquad k_B T(L_\kappa) = \frac{\hbar c}{2\pi L_\kappa}.} \tag{70}$$

The first is classical gravitational boundary mechanics and scales as L_κ^{-2}; the second is semiclassical quantum-field temperature and scales as L_κ^{-1}. Their ratio is independent of the clock scale:

$$\boxed{\frac{k_B T}{P_L L_\kappa} = 4\ell_{\mathrm{P}}^2.} \tag{71}$$

The numerical entropy-density relation could already be written in the established membrane form $\Pi_\partial/T = S/A$. The acceleration-length factorization

$$\Pi_\partial = P_L L_\kappa, \qquad P_L = -\left(\frac{\partial|\Pi_\partial|}{\partial L_\kappa}\right) \tag{72}$$

does not alter that coefficient; it supplies the missing mechanical resolution. It identifies a normal force $F_L = AP_L$, a displacement L_κ, and pressure work $F_L\, dL_\kappa = P_L\, \vartheta_V$. It also makes the same factorization responsible for the Schwarzschild scale product and the Cai–Kim FRW work term. The entropy density can therefore be read macroscopically as

$$\frac{S}{A} = \frac{P_L L_\kappa}{T}: \qquad \text{mechanical boundary-energy density divided by temperature.} \tag{73}$$

This is the interpretive gain beyond substituting the established surface tension into the Hawking relation.

The boundary potential is separately homogeneous of degree one in A and degree minus one in L_κ:

$$A\left(\frac{\partial U_\partial}{\partial A}\right)_{L_\kappa} = -L_\kappa\left(\frac{\partial U_\partial}{\partial L_\kappa}\right)_A = U_\partial. \tag{74}$$

Its integrated representations form the Euler chain

$$\boxed{U_\partial = TS = \Pi_\partial A = P_L(AL_\kappa) = F_L L_\kappa.} \tag{75}$$

Combining Eqs. (71) and (75) gives

$$\frac{S}{A} = \frac{P_L L_\kappa}{T} = \frac{k_B}{4\ell_{\mathrm{P}}^2}. \tag{76}$$

As $|\kappa| \to 0$, both $P_L L_\kappa = c^2|\kappa|/(8\pi G)$ and $T = \hbar|\kappa|/(2\pi k_B c)$ vanish linearly, so their ratio retains the finite branchwise limit in Eq. (76). For a regular finite-area Einstein horizon, TS/A can therefore vanish while S/A remains finite. The compatibility of zero temperature with finite entropy is familiar; the pressure–length factorization makes the matching mechanical and thermal scalings explicit. At the exact zero, the reciprocal chart is unavailable and the statement is understood as a limit from either nonzero branch.

The factor $1/4$ is fixed by the ratio of the Einstein boundary coefficient $1/(8\pi G)$ to the Hawking–Unruh coefficient $1/(2\pi)$; surface gravity and clock normalization cancel. This is not a purely general-relativistic derivation because the temperature contains $\hbar$. It is a complete macroscopic boundary-mechanical reconstruction of the Einstein-horizon entropy density at the interface of classical gravitational action and semiclassical quantum-field temperature. It is a consistency reconstruction using the Einstein entropy law already stated in Eq. (63). If one instead starts with $T\,dS = \Pi_\partial\, dA$, integration determines the area term plus an additive constant S_0; the Euler equality $U_\partial = TS$ uses the conventional $S_0 = 0$ normalization. No microscopic state count or independent entropy functional is derived.

Ordinary state counting is not a prerequisite for treating this as real entropy. Wald emphasizes that the classical and semiclassical evidence supports black-hole entropy while “it does not seem obvious that black hole entropy corresponds to counting states in any usual sense” [8, 9]. One possible macroscopic informational interpretation is a horizon supplies an observer- or evolution-relative partition between accessible and inaccessible data; boundary mechanics fixes the energy density assigned to that partition; and temperature converts it into entropy. The claim is not that S inventories every detailed fact in the hidden region. Such a reading would relate it to information unavailable to, or thermodynamically indistinguishable within, the selected exterior description. The present equations do not select an observable algebra, coarse-graining map, or statistical entropy whose equality to the area law has been proved.

For a black-hole event horizon the partition is causal for the exterior observer. For an eternally accelerated Rindler observer it is causal and permanent for that observer but remains observer-relative because inertial observers do not share it. A generic FRW apparent horizon is a quasi-local marginal screen and need not be a permanent causal barrier; exact de Sitter is the special case in which the apparent and cosmological event horizons coincide.

At fixed area the differential identity is

$$-S\,dT = F_L\,dL_\kappa = P_L\,\vartheta_V. \tag{77}$$

Gravitational cooling and outward scale work are the same canonical variation in this ensemble. This does not by itself specify an extractable-work protocol.

With

$$\mathcal{V}_\partial \equiv AL_\kappa, \qquad N_S \equiv \frac{S}{k_B}, \tag{78}$$

Eq. (75) becomes

$$\boxed{P_L\mathcal{V}_\partial = N_S k_B T = U_\partial.} \tag{79}$$

This exact ideal-gas-form Euler relation follows from boundary homogeneity, not from assumed molecules. Here $\mathcal{V}_\partial$ is the product AL_κ; its full differential is Eq. (51), not the normal-scale one-form ϑ_V.

a. Spherical gas form and the choice of energy. On the positive spherical families with fixed $\alpha_R = L_\kappa/R$, Sec. V G gives $P_R = \alpha_R P_L$ and $U_\partial = 3P_R V = TS$. Consequently,

$$P_R V = N_{\rm eff} k_B T, \qquad U_\partial = 3N_{\rm eff} k_B T, \qquad N_{\rm eff} \equiv \frac{S}{3k_B}. \tag{80}$$

This pair has the algebraic form of a three-dimensional classical ultrarelativistic ideal gas: a Maxwell–Boltzmann gas with single-particle energy $\varepsilon \simeq pc$ has a one-particle partition function proportional to VT^3, giving $pV = Nk_BT$ and $U_{\rm gas} = 3Nk_BT$ [66]. Here "classical" specifies the statistics, not a nonrelativistic dispersion relation. Equation (80) is a macroscopic representation with an entropy-defined $N_{\rm eff}$; it neither assumes nor excludes a microscopic realization. The thermodynamic status of horizon entropy does not depend on first identifying molecules or proving an ordinary state count [8]. What remains open is whether a statistical model reproduces this particular effective count and the full boundary responses. In particular, $S \propto R^2$ and $V \propto R^3$ are linked along the horizon family, whereas a gas can vary its volume independently of its entropy and particle number.

Blackbody radiation supplies a related but different comparison. Let U_γ be the internal energy in the photons, excluding the cavity and its mechanical load. The extensive equilibrium Euler identity is $TS = U + pV - \mu N$; photon-number equilibrium sets $\mu_\gamma = 0$. Isotropic radiation has $p_\gamma V_\gamma = U_\gamma/3$, so

$$\mathcal{H}_\gamma \equiv T_\gamma S_\gamma = U_\gamma + p_\gamma V_\gamma = \frac{4}{3}U_\gamma = 4p_\gamma V_\gamma. \tag{81}$$

The energy-valued product $T_\gamma S_\gamma$ is therefore the photon *enthalpy*, not its internal energy [67]. Comparing like thermal products gives $U_\partial = TS = 3P_R V$ for the spherical horizon and $\mathcal{H}_\gamma = T_\gamma S_\gamma = 4p_\gamma V_\gamma$ for radiation. Renaming the latter product does not remove the distinction, because its differential is

$$dU_\gamma = T_\gamma dS_\gamma - p_\gamma dV_\gamma, \qquad d\mathcal{H}_\gamma = T_\gamma dS_\gamma + V_\gamma dp_\gamma. \tag{82}$$

One cannot apply the ordinary extensive Euler relation to a constrained gravitational boundary potential simply because it also has energy units. The relevant horizon homogeneity is Eq. (74) and, for spherical scaling, Eq. (150). Conversely, the formal rescaling $E_{\rm match} = 3U_\partial/4$ makes $TS = 4E_{\rm match}/3$, but also gives $dE_{\rm match} = (3/4)T\,dS - (3/4)F_L\,dL_\kappa$. Matching the thermal product alone does not preserve the first-law coefficients or identify this rescaled quantity as photon internal energy.

The process must also be specified. A reversible adiabatic expansion of radiation in a movable cavity gives $dU_\gamma = -p_\gamma dV_\gamma$, hence $U_\gamma \propto V_\gamma^{-1/3}$. The photons transfer energy to the moving boundary as work; including the boundary and load restores the conserved energy of the isolated whole. Adiabatic means no heat transfer, not no energy transfer [68]. This is not the changing-horizon path, for which $U_\partial \propto R$ and the area entropy changes.

These spherical comparisons do not replace the Rindler relation (79). For a finite planar patch, AL_κ can be realized as the volume of a specified slab on the instantaneous rest slice, and $d(AL_\kappa) = A\,dL_\kappa$ at fixed transverse area. It is not a universal enclosing volume or an assertion that the slab is filled with material carrying the boundary energy. If area varies, the additional term $L_\kappa\,dA$ must be retained. Acceleration-length and radius-volume representations therefore remain complementary.

b. Boundary inverse-energy length, phase rate, and entropy gradient. The boundary potential also defines a quantum inverse-energy scale. Set

$$M_\partial \equiv \frac{U_\partial}{c^2}, \qquad \bar{\lambda}_\partial \equiv \frac{\hbar}{M_\partial c} = \frac{\hbar c}{U_\partial}. \tag{83}$$

For a particle whose M_∂ is invariant rest mass, this is the ordinary reduced Compton wavelength. Here M_∂ is instead the mass equivalent of a clock-normalized boundary potential, not generally particle rest mass, ADM mass, or source mass. The more precise name is therefore *boundary Compton-equivalent length*.

The inverse-energy definition is exact; the phase interpretation is conditional. In any quantum realization in which U_∂ is an eigenvalue, or a sharply defined expectation value, of the Hamiltonian that generates the selected boundary time, Schrödinger evolution gives

$$\omega_\partial \equiv \frac{U_\partial}{\hbar}, \qquad \bar{\lambda}_\partial = \frac{c}{\omega_\partial}. \tag{84}$$

Then $\bar{\lambda}_\partial$ is the light-travel distance associated with one radian of the corresponding quantum phase. The present macroscopic analysis does not construct a boundary Hilbert space or prove that U_∂ is a sharp Hamiltonian eigenvalue. Accordingly, $\bar{\lambda}_\partial$ is rigorously a boundary inverse-energy length; the Compton-equivalent phase language records the standard energy–frequency map and does not assert a propagating boundary particle or a directly observable global oscillation.

The ratio raised naturally by the boundary first law has a direct thermodynamic meaning. At fixed U_∂,

$$\frac{T}{F_L} = \left(\frac{\partial L_\kappa}{\partial S}\right)_{U_\partial} = \frac{L_\kappa}{S} = \frac{\bar{\lambda}_\partial}{2\pi k_B}, \tag{85}$$

so T/F_L is length per entropy, while

$$\boxed{\frac{k_B T}{F_L} = \frac{L_\kappa}{N_S} = \frac{\bar{\lambda}_\partial}{2\pi}.} \tag{86}$$

The quantity $k_B T/F_L$ is the displacement over which the boundary force does one thermal energy $k_B T$ of work. Equivalently,

$$\boxed{2\pi L_\kappa = N_S \bar{\lambda}_\partial, \qquad F_L \bar{\lambda}_\partial = 2\pi k_B T.} \tag{87}$$

Let

$$C_E \equiv c\Delta\tau_E = 2\pi L_\kappa, \qquad \lambda_\partial \equiv 2\pi\bar{\lambda}_\partial \tag{88}$$

denote the Euclidean thermal circumference in light-travel units and the corresponding full boundary Compton-equivalent wavelength. Then

$$\boxed{\frac{C_E}{\bar{\lambda}_\partial} = N_S, \qquad \frac{C_E}{\lambda_\partial} = \frac{N_S}{2\pi}, \qquad \frac{\omega_\partial}{\omega_1^{\mathrm{E}}} = \frac{N_S}{2\pi}.} \tag{89}$$

Because $\bar{\lambda}_\partial$ is a reduced, one-radian inverse-energy length, the first ratio counts such reduced intervals and the second counts full 2π intervals. If a quantum boundary Hamiltonian realizes U_∂, these are respectively phase radians and full phase cycles. Without that additional microscopic realization, the exact statement is the ratio of the Euclidean thermal period to the inverse-energy time $\hbar/U_\partial$. Analytically continued around one thermal period, the same statement is

$$\boxed{\frac{U_\partial \Delta\tau_E}{\hbar} = \beta U_\partial = \frac{S}{k_B} = N_S.} \tag{90}$$

This is the dimensionless Euclidean Boltzmann exponent associated with the boundary potential over one thermal period. Equivalently, S/k_B is the boundary energy measured in thermal units k_BT. If U_∂ is realized as a quantum time-translation generator, Wick rotation turns its Lorentzian phase factor into this Euclidean decay exponent, so the same quantity may also be read as the boundary-energy phase advance continued around the thermal circle. It is not a claim that βU_∂ equals the complete on-shell gravitational action or that the horizon is literally tiled by independent Compton cells. The model-independent content is that entropy measures the separation between the boundary inverse-energy scale and the Euclidean thermal scale.

At fixed U_∂, Eq. (85) may also be written

$$dS = 2\pi k_B \frac{dL_\kappa}{\bar{\lambda}_\partial}. \tag{91}$$

A displacement by one boundary Compton-equivalent reduced length gives $\Delta S = 2\pi k_B$. Verlinde postulated the analogous relation for the displacement of an external test particle relative to a holographic screen, motivated by Bekenstein's lowering argument, and chose the 2π normalization so that it combines with Unruh temperature to give $F = ma$ [3]. Equation (91) has the same mathematical form but a different status and meaning: Δx is replaced by the boundary source displacement dL_κ, the particle mass is replaced by the boundary mass equivalent, and the coefficient follows from the Einstein boundary potential and its Hamilton–Jacobi force. This supports a genuine horizon-boundary origin for the displacement law but does not establish the broader claim that all Newtonian gravity is an entropic force or justify arbitrary holographic screens and equipartition.

The integrated relation is

$$\frac{S}{k_B} = \frac{2\pi U_\partial L_\kappa}{\hbar c} = \frac{2\pi L_\kappa}{\bar{\lambda}_\partial}. \tag{92}$$

It has the algebraic form of saturation of the Bekenstein entropy bound [2]. For Schwarzschild, $U_\partial = Mc^2/2$ and $L_\kappa = 2R_H$, so the product is the usual $Mc^2 R_H$; for Cai–Kim FRW, $U_\partial = E_A$ and $L_\kappa = R_A$. For a generic boundary sector, however, U_∂ need not be the total system energy and L_κ need not be an enclosing radius. The result is therefore a horizon-boundary equality of Bekenstein form: it suggests that the normal-flow product $U_\partial L_\kappa$, rather than an enclosing radius alone, is the more general boundary object. It is not a new proof of the Bekenstein bound for arbitrary systems.

The ratio between the quantum and horizon scales is especially simple:

$$\boxed{\frac{\bar{\lambda}_\partial}{L_\kappa} = \frac{2\pi}{N_S} = \frac{8\pi\ell_{\mathrm{P}}^2}{A}.} \tag{93}$$

It is clock invariant and controlled only by area in Planck units. Macroscopic horizons have $\bar{\lambda}_\partial \ll L_\kappa$: the boundary potential is many thermal units k_BT, and the inverse-energy time $\hbar/U_\partial$ is much shorter than the Euclidean thermal period. If the quantum phase interpretation applies, this is equivalently $\omega_\partial \gg \omega_1^{\mathrm{E}}$ and many phase cycles fit around the Euclidean thermal circle. At

$$A = 8\pi\ell_{\mathrm{P}}^2, \qquad S = 2\pi k_B, \tag{94}$$

the two reduced lengths and the two frequencies coincide:

$$\boxed{\bar{\lambda}_\partial = L_\kappa, \qquad \omega_\partial = \omega_1^{\mathrm{E}}, \qquad U_\partial = \hbar\omega_1^{\mathrm{E}} = 2\pi k_B T.} \tag{95}$$

Equivalently, one full boundary Compton-equivalent wavelength $2\pi\bar{\lambda}_\partial$ fits around the Euclidean thermal circumference $2\pi L_\kappa$. This is a natural quantum–horizon crossover benchmark: the inverse-energy scale of the complete boundary potential is no longer parametrically separated from its geometric and thermal scale. It is not evidence by itself for a phase transition, but it marks the loss of the large-entropy hierarchy on which a semiclassical description is normally most persuasive.

For Schwarzschild, $M_\partial = M/2$, so $\bar{\lambda}_\partial = 2\hbar/(Mc)$, while $L_\kappa = 2R_H$. Consequently $\bar{\lambda}_\partial = L_\kappa$ is exactly equivalent to the conventional reduced-Compton condition $\hbar/(Mc) = R_H$. With the standard definitions this occurs at

$$M = \frac{m_{\mathrm{P}}}{\sqrt{2}}, \qquad R_H = \sqrt{2}\,\ell_{\mathrm{P}}, \qquad L_\kappa = \bar{\lambda}_\partial = 2\sqrt{2}\,\ell_{\mathrm{P}}. \tag{96}$$

The familiar statement that the Compton and Schwarzschild scales meet "at the Planck scale" therefore carries order-one convention factors: using the gravitational radius GM/c^2 instead of $R_H = 2GM/c^2$, or the ordinary rather than reduced Compton wavelength, moves the numerical crossing. The present boundary variables reproduce the conventional reduced-Compton–Schwarzschild-radius crossing because both lengths are doubled.

c. Planck units and the status of ultraviolet benchmarks. The Planck quantities used here are natural combinations of the constants that normalize relativity, quantum mechanics, and classical gravity:

$$\ell_{\rm P} = \sqrt{\frac{\hbar G}{c^3}}, \qquad t_{\rm P} = \frac{\ell_{\rm P}}{c}, \qquad m_{\rm P} = \sqrt{\frac{\hbar c}{G}}, \qquad E_{\rm P} = m_{\rm P} c^2, \tag{97}$$

$$a_{\rm P} \equiv \frac{c}{t_{\rm P}} = \frac{c^2}{\ell_{\rm P}}, \qquad F_{\rm P} \equiv \frac{E_{\rm P}}{\ell_{\rm P}} = \frac{c^4}{G}. \tag{98}$$

They are not arbitrary symbols: for example, the dimensionless quantum-gravitational coupling constructed from an energy scale E is

$$\alpha_G(E) \equiv \frac{GE^2}{\hbar c^5} = \left(\frac{E}{E_{\rm P}}\right)^2, \tag{99}$$

so the Planck energy is where this simple coupling estimate becomes order unity. The definitions do not establish that $\ell_{\rm P}$ is an exact minimum length, that $a_{\rm P}$ is a maximum acceleration, or that the unmodified Einstein and Hawking coefficients remain exact at those scales. General relativity is a controlled quantum effective field theory well below the Planck scale, while its ultraviolet completion is unknown [5]. The relations below are consequently benchmarks obtained by propagating the present Einstein boundary and Hawking–Unruh normalizations, not predictions of completed quantum gravity.

The Rindler relation $L_\kappa = c^2/a$ is exact within classical Minkowski geometry. Inserting $a = a_{\rm P}$ gives $L_\kappa = \ell_{\rm P}$ by definition. This does not demonstrate a realizable Planck-accelerated detector or a universal acceleration bound; it identifies the point at which the classical Rindler proper distance reaches the Planck length.

d. A hierarchy of distinct Planck benchmarks. The least restrictive and most clock-invariant criterion internal to the present framework is the loss of scale separation,

$$\boxed{\bar\lambda_\partial = L_\kappa.} \tag{100}$$

It fixes the transverse data

$$A = 8\pi\ell_{\rm P}^2, \qquad R = \sqrt{2}\,\ell_{\rm P} \quad \text{for a spherical cut}, \tag{101}$$

but it does not fix the common value of $\bar\lambda_\partial$ and L_κ. The reason is that a constant rescaling of the selected normal clock rescales U_∂, $\bar\lambda_\partial$, and L_κ together while leaving A unchanged. A physical sector supplies the remaining relation: Cai–Kim gives $L_\kappa = R = \sqrt{2}\,\ell_{\rm P}$, whereas Schwarzschild gives $L_\kappa = 2R_H = 2\sqrt{2}\,\ell_{\rm P}$ at the same crossover area. The area is common; the normal geometry is sector dependent.

A stronger, maximally symmetric source-space benchmark adds the Planck normal scale,

$$\boxed{L_\kappa = \bar\lambda_\partial = \ell_{\rm P}.} \tag{102}$$

Equations (94) and (102) then give

$$\boxed{A = 8\pi\ell_{\rm P}^2, \quad R = \sqrt{2}\,\ell_{\rm P}, \quad U_\partial = E_{\rm P}, \quad M_\partial = m_{\rm P}, \quad F_L = F_{\rm P}, \quad k_BT = \frac{E_{\rm P}}{2\pi}.} \tag{103}$$

This is the simplest all-Planck alignment in the extended (A, L_κ) boundary-source space. It is not the conventional Schwarzschild crossover and it is not the Cai–Kim relation $L_\kappa = R$: for a spherical cut it has $R/L_\kappa = \sqrt{2}$.

A different, equally simple but sector-specific benchmark is to impose a Planck areal radius in the Cai–Kim relation,

$$R = L_\kappa = \ell_{\rm P}. \tag{104}$$

Keeping the standard Einstein normalization then yields

$$\boxed{A = 4\pi\ell_{\rm P}^2, \quad U_\partial = \frac{E_{\rm P}}{2}, \quad M_\partial = \frac{m_{\rm P}}{2}, \quad F_L = \frac{F_{\rm P}}{2}, \quad \bar\lambda_\partial = 2\ell_{\rm P}, \quad S = \pi k_B.} \tag{105}$$

There is no inconsistency: the condition $R = \ell_{\rm P}$ fixes only half of the crossover area. Requiring in addition $U_\partial = E_{\rm P}$ and $F_L = F_{\rm P}$ at $R = L_\kappa = \ell_{\rm P}$ would double the Einstein boundary coefficient, equivalently replacing G by an effective $G/2$ in this one-scale benchmark. No established Planck-regime principle presently selects that modification.

The hierarchy adopted here is therefore: use $\bar\lambda_\partial = L_\kappa$ as the primary quantum–horizon crossover because it compares the two scales defined internally by the framework; use $L_\kappa = \bar\lambda_\partial = \ell_{\rm P}$ as the especially symmetric all-Planck benchmark; and treat $R = L_\kappa = \ell_{\rm P}$ as a separate Cai–Kim-like Planck-radius continuation. None is claimed to be the unique state chosen by quantum gravity.

F. Compliance, modulus, and source-space Hessian

The inverse pressure derivative defines a source compliance. At fixed transverse area,

$$\mathcal{V}_\partial = AL_\kappa, \qquad P_L = \frac{c^4}{8\pi G L_\kappa^2} = \frac{c^4 A^2}{8\pi G \mathcal{V}_\partial^2}. \tag{106}$$

Therefore

$$\boxed{\left(\frac{\partial \mathcal{V}_\partial}{\partial P_L}\right)_A = -\frac{\mathcal{V}_\partial}{2P_L}.} \tag{107}$$

The associated scale compressibility and bulk modulus are

$$\chi_L^{(A)} \equiv -\frac{1}{\mathcal{V}_\partial}\left(\frac{\partial \mathcal{V}_\partial}{\partial P_L}\right)_A = \frac{1}{2P_L}, \qquad B_L^{(A)} \equiv -\mathcal{V}_\partial\left(\frac{\partial P_L}{\partial \mathcal{V}_\partial}\right)_A = 2P_L. \tag{108}$$

The positive modulus means that increasing the normal scale lowers its conjugate pressure at fixed area. It is a source susceptibility; a restoring motion or stability criterion additionally requires a load, an equation of motion, and boundary conditions.

Using Eq. (70),

$$P_L(T) = \frac{\pi c^2 k_B^2}{2G\hbar^2}T^2, \qquad \frac{dP_L}{dT} = \frac{2P_L}{T}, \tag{109}$$

and at fixed area

$$\mathcal{V}_\partial T = \frac{A\hbar c}{2\pi k_B}, \qquad \alpha_L^{(A)} \equiv \frac{1}{\mathcal{V}_\partial}\left(\frac{\partial \mathcal{V}_\partial}{\partial T}\right)_A = -\frac{1}{T}. \tag{110}$$

These are response functions of the acceleration-length representation.

The Hessian of $U_\partial = \mathcal{T}_G A/L_\kappa$ on the extended source space is

$$\frac{\partial^2 U_\partial}{\partial(A, L_\kappa)^2} = \begin{pmatrix} 0 & -P_L \\ -P_L & 2F_L/L_\kappa \end{pmatrix}, \qquad \det = -P_L^2 < 0. \tag{111}$$

The potential is a saddle if A and L_κ are treated as independent sources. This is not by itself a dynamical instability because the variables define a boundary ensemble, not an unconstrained material state space. The positive fixed-area modulus in Eq. (108) is the relevant local scale response.

V. STATIONARY REALIZATIONS AND THERMAL–MECHANICAL DUALITY

A. Rindler: the operational meaning of acceleration length

For an observer with constant proper acceleration a in Minkowski spacetime, introduce coordinates centered on that observer,

$$cT = \left(\frac{c^2}{a} + \xi\right)\sinh\left(\frac{at}{c}\right), \qquad X = \left(\frac{c^2}{a} + \xi\right)\cosh\left(\frac{at}{c}\right). \tag{112}$$

The observer is at $\xi = 0$ and the Rindler horizon is at $\xi = -c^2/a$. The instantaneous rest slice therefore meets the horizon a proper distance

$$L_\kappa = \frac{c^2}{a} \tag{113}$$

behind the observer. The Lorentzian trajectory satisfies $X^2 - c^2T^2 = L_\kappa^2$, a hyperbola. Under the analytic continuation $t = -i\tau_E$ it becomes

$$cT_E = L_\kappa \sin\left(\frac{a\tau_E}{c}\right), \qquad X = L_\kappa \cos\left(\frac{a\tau_E}{c}\right), \tag{114}$$

a Euclidean circle of radius L_κ. The Rindler horizon $\rho = 0$ becomes the origin of this normal Euclidean plane; in four dimensions the transverse horizon plane remains as the fixed-point set. One smooth circuit has $a\Delta\tau_E/c = 2\pi$, reproducing the Unruh period. Rindler therefore makes L_κ simultaneously the observer–horizon proper distance, the curvature radius of the accelerated worldline, the Euclidean orbit radius, and the reduced thermal length [7, 35, 36].

Proper time fixes the physical clock and $\kappa = a$. The source proposition and corner action then give

$$\boxed{P_L^{\mathrm{R}} = \frac{a^2}{8\pi G}, \qquad F_L^{\mathrm{R}} = \frac{Aa^2}{8\pi G}, \qquad U_\partial^{\mathrm{R}} = \frac{Aac^2}{8\pi G}.} \tag{115}$$

No source mass, curvature, center, or spherical volume is used. To make the transverse geometry explicit, choose Cartesian coordinates y, z on a horizon cut and a finite rectangle $\mathcal{D}$. Then

$$A_{\mathcal{D}} = \int_{\mathcal{D}} dy\, dz = \Delta y\, \Delta z, \qquad U_{\partial,\mathcal{D}} = \frac{A_{\mathcal{D}} ac^2}{8\pi G}. \tag{116}$$

Changing the selected patch at fixed acceleration changes U_∂ and F_L in proportion to area, but changes neither L_κ nor P_L. No projection of a sphere is involved. This is the cleanest local demonstration that the normal pressure is not merely a Newtonian source-field formula in disguise.

For an observer who accelerates uniformly for all proper time, the Rindler horizon is a permanent causal barrier for that observer. It is nevertheless observer-relative: inertial observers in the same Minkowski spacetime do not share the partition, and a finite-duration accelerated trajectory need not possess the same eternal event horizon. The nonzero response in flat spacetime is therefore attached to the accelerated observer's boundary and clock, not to an observer-independent stress of the Minkowski vacuum.

Frodden, Ghosh, and Perez considered stationary observers at a small but finite proper distance ℓ outside a nonextremal horizon, with ℓ much smaller than the horizon curvature scale. At leading near-horizon order they found the local energy

$$E_{\mathrm{FGP}} \simeq \frac{c^4 A}{8\pi G\ell}. \tag{117}$$

Their locally normalized surface gravity satisfies $\kappa_{\mathrm{loc}} \simeq c^2/\ell$, hence $L_{\mathrm{loc}} = c^2/\kappa_{\mathrm{loc}} \simeq \ell$ in the same approximation [18]. This identifies a close local antecedent of the potential; it does not identify ℓ with every horizon's acceleration length. In Schwarzschild, for example, a static observer at radius $r > R_H$ has lapse $N(r) = \sqrt{1 - R_H/r}$ relative to infinity, so

$$\kappa_{\mathrm{loc}} = \frac{\kappa_\infty}{N(r)}, \qquad L_{\mathrm{loc}} = N(r)L_\infty = 2R_H N(r) \simeq \ell, \qquad L_\infty = 2R_H. \tag{118}$$

The last approximate equality is restricted to the near-horizon region; in exact Rindler geometry the distance equality is exact.

FGP's local first law varies the horizon state at prescribed observer distance, giving $\delta E_{\mathrm{FGP}} \simeq [c^4/(8\pi G\ell)]\delta A$. Moving the observer by varying ℓ on a fixed black-hole geometry also changes the local clock normalization. Differentiating the leading A/ℓ expression then gives a candidate distance response, but does not establish the same variational protocol as changing the mixed gravitational rate source with its boundary data specified. The present contribution is the branchwise source $L_s = c^2/\kappa$, its canonical conjugate response in that ensemble, and the sector-dependent geometric and thermal realizations. Neither the finite size of L_s nor the small size of ℓ alone establishes or excludes a work-coordinate interpretation.

At fixed area the leading FGP expression has the explicit derivative

$$-\left(\frac{\partial E_{\mathrm{FGP}}}{\partial \ell}\right)_A \simeq \frac{Ac^4}{8\pi G\ell^2}. \tag{119}$$

The inverse-square coefficient is therefore already present locally. The distinction is the protocol: the FGP area law compares horizon states at a prescribed observer-distance normalization, while a distance variation also changes that observer family and clock. The null source calculation specifies the gravitational boundary data of its variation. The two descriptions overlap after matching them; neither distance smallness nor a formal negative derivative alone proves a material work-extraction law. Away from the horizon, redshifted horizon surface gravity also differs from the static observer's proper acceleration.

B. Spherical completion of Rindler boundary data

The Rindler distance $L_\kappa = c^2/a$ is a normal distance. Changing the transverse area means choosing a larger or smaller region of the same planar horizon cut, while retaining the same acceleration and clock. It does not move the horizon closer to or farther from the observer. The local uniform boundary data (A, L_κ) therefore remain independent. A sphere introduces additional transverse information.

There is nonetheless an exact and useful spherical completion of these data. Choose the positive branch and set

$$R_{\rm eq} = L_\kappa, \qquad A_{\rm eq} = 4\pi L_\kappa^2, \qquad V_{\rm eq} = \frac{4\pi}{3} L_\kappa^3. \tag{120}$$

Evaluating the same Einstein boundary potential and normal pressure on this one-parameter family gives

$$\boxed{U_{\rm eq} = \frac{c^4 L_\kappa}{2G}, \qquad F_{L,\rm eq} = \frac{c^4}{2G}, \qquad \rho_{\rm eq} \equiv \frac{U_{\rm eq}}{V_{\rm eq}} = \frac{3c^4}{8\pi G L_\kappa^2} = 3P_L.} \tag{121}$$

The associated mass equivalent obeys

$$M_{\rm eq} \equiv \frac{U_{\rm eq}}{c^2} = \frac{c^2 R_{\rm eq}}{2G}, \qquad R_{\rm eq} = \frac{2GM_{\rm eq}}{c^2}. \tag{122}$$

Here $F_{L,\rm eq}$ means the independent-source force evaluated on the chosen family. It is not defined by the negative total derivative of $U_{\rm eq}$ along that family: the area changes as well. Indeed,

$$\Pi_\partial dA_{\rm eq} = 2\, dU_{\rm eq}, \qquad F_{L,\rm eq}\, dL_\kappa = dU_{\rm eq}, \qquad dU_{\rm eq} = \Pi_\partial dA_{\rm eq} - P_L dV_{\rm eq}. \tag{123}$$

With the usual thermal identification, the last equation is exactly the Cai–Kim screen-state law. Thus the energy, pressure, and complete state differential all match, not merely the dimensions. The match follows from the common boundary potential after the particular relation $A = 4\pi L_\kappa^2$ is imposed. The constructed $\rho_{\rm eq}$ is not thereby a bulk density in Minkowski spacetime.

The geometrical distinction can be stated invariantly. In a spherically symmetric spacetime $ds^2 = h_{ab}dx^a dx^b + R^2 d\Omega_2^2$, the Misner–Sharp energy is

$$E_{\rm MS} = \frac{c^4 R}{2G}\left(1 - h^{ab}\partial_a R\, \partial_b R\right). \tag{124}$$

A spherical marginal surface has $h^{ab}\partial_a R \partial_b R = 0$, so $R = 2GM_{\rm MS}/c^2$, with $M_{\rm MS} = E_{\rm MS}/c^2$ [62, 63]. This is the genuine common mass–radius relation for Schwarzschild and FRW apparent horizons. For round spheres in Minkowski spacetime the contraction is one and $E_{\rm MS} = 0$. Changing observers or multiplying a Rindler patch's boundary-energy density by a chosen area does not change that result.

Nor does a radial set of accelerated observers turn flat spacetime into FRW. A single observer has one proper-acceleration vector; a radially arranged congruence introduces many observers and extra geometric choices. Spherical Rindler coordinates illustrate that such congruences can exist in flat spacetime, with a spherical size parameter independent of the local acceleration-distance coordinate [64, 65]. The timelike FRW comoving congruence, by contrast, has zero proper acceleration. Its apparent horizon is fixed by the spacetime's expansion and spherical null expansions. Consequently the identification $a = c^2/R_A$ is a matching of normal boundary scales, not an identification of comoving proper acceleration with horizon surface gravity.

De Sitter is the clean exact spherical realization of the matching data: its cosmological horizon has $R_{\rm dS} = c/H$, $|\kappa| = cH = c^2/R_{\rm dS}$, and $E_{\rm MS} = c^4 R_{\rm dS}/(2G)$. With a positive-rate orientation its boundary potential equals this energy. With the signed cosmological Kodama convention the potential is negative and its magnitude equals the same energy. The stationary horizon also has $T = \hbar H/(2\pi k_B)$, although its central comoving observer is geodesic [48]. Generic Cai–Kim FRW preserves the instantaneous $L_\kappa = R_A$ calibration, without requiring a stationary or global event horizon.

C. Why the common mass–radius relation does not fix acceleration length

For a static spherical line element

$$ds^2 = -f(r)c^2 dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \qquad |\kappa| = \frac{c^2}{2}|f'(R)|, \tag{125}$$

where the horizon is a simple root $f(R) = 0$, the clock is that of the displayed t. Standard infinity-normalized Schwarzschild and center-normalized de Sitter give, respectively,

$$f_{\rm Schw}(r) = 1 - \frac{R_H}{r}, \qquad f'_{\rm Schw}(R_H) = \frac{1}{R_H}, \qquad L_\kappa^{\rm Schw} = 2R_H,$$
$$f_{\rm dS}(r) = 1 - \frac{r^2}{R_{\rm dS}^2}, \qquad f'_{\rm dS}(R_{\rm dS}) = -\frac{2}{R_{\rm dS}}, \qquad L_\kappa^{\rm dS} = R_{\rm dS}. \tag{126}$$

Both horizons are locally Rindler. Their different acceleration-length ratios are controlled by the radial metric gradients and selected normalizations; a local Rindler approximation does not fix those global data. The horizon mass–radius relation fixes $f(R) = 0$, whereas surface gravity also requires $f'(R)$. This is the geometrical reason that a shared $R = 2GM_{\rm MS}/c^2$ need not imply a shared L_κ/R. For the metric in Eq. (125), writing $f(r) = 1 - 2Gm(r)/(c^2 r)$ makes the distinction explicit:

$$\kappa_s = \frac{c^2}{2R} - \frac{Gm'(R)}{R}, \qquad m(R) = \frac{c^2 R}{2G}. \tag{127}$$

Schwarzschild has $m' = 0$; de Sitter has a volume-proportional mass profile. More general static lapse functions require their own normalization factors. Neither $L_\kappa = R$ nor $L_\kappa = 2R$ is a universal inference from the horizon's mass–radius relation alone.

D. Schwarzschild pressure and field stress

Let $R_H = 2GM/c^2$ and normalize the stationary Killing field to unit time at infinity. Then

$$\kappa_\infty = \frac{GM}{R_H^2} = \frac{c^2}{2R_H} = \frac{c^4}{4GM}, \qquad L_\kappa = 2R_H. \tag{128}$$

With $A_H = 4\pi R_H^2$,

$$\boxed{P_L^{\rm Schw} = \frac{c^4}{32\pi G R_H^2}, \qquad F_L^{\rm Schw} = \frac{c^4}{8G}, \qquad U_\partial^{\rm Schw} = F_L L_\kappa = TS = \frac{Mc^2}{2}.} \tag{129}$$

The force is independent of black-hole size in four spacetime dimensions. It is conjugate to $L_\kappa = 2R_H$, not directly to R_H. For the conventional areal volume $V_H = 4\pi R_H^3/3$,

$$F_L\, dL_\kappa = 2F_L\, dR_H = (M\kappa_\infty)\, dR_H = \frac{M\kappa_\infty}{A_H}\, dV_H = 2P_L\, dV_H. \tag{130}$$

The radius-work force is $F_R^{(L)} = 2F_L = M\kappa_\infty = c^4/(4G)$, and its force per area is $2P_L$. Thus the mass-times-surface-gravity interpretation follows exactly for this scale leg of dU_∂. The full area term remains: $dU_\partial = \Pi_\partial dA_H - (M\kappa_\infty)dR_H$ with $U_\partial = Mc^2/2$. It is not an additional work term in the separate mass law $d(Mc^2) = TdS$. The canonical values conjugate to L_κ remain F_L and P_L. Nor does $L_\kappa = 2R_H$ make L_κ a proper radial distance on an arbitrary spatial slice.

There is also a suggestive semiclassical scale correspondence. In geometric units $G = c = 1$, $R_H = 2M$, $\kappa = 1/(4M)$, and $L_\kappa = 4M$. For a freely falling observer, Dey, Liberati, Mirzaiyan, and Pranzetti write the WKB adiabaticity parameter of an outgoing mode as

$$\delta_\omega = \frac{1}{4M\omega_0} V_r^3 (1 - V_r^2), \qquad V_r = \sqrt{\frac{2M}{r}}. \tag{131}$$

Writing $y \equiv V_r^2 = 2M/r = L_\kappa/(2r)$ makes the scale structure explicit:

$$\delta_\omega = \frac{1}{\omega_0 L_\kappa} y^{3/2}(1 - y). \tag{132}$$

The factor $y^{3/2}$ grows inward as the free-fall and geometric-variation scale strengthens, whereas $1 - y = 1 - 2M/r$ is the Schwarzschild lapse factor and drives this particular freely falling WKB diagnostic back to zero at the horizon.

The competition therefore has a maximum between the weak-field exterior and the horizon. Maximizing $y^{3/2}(1-y)$ gives $y = 3/5$ and hence

$$r_{\mathrm{WKB}} = \frac{10M}{3} = \frac{5}{6}L_\kappa. \tag{133}$$

The coefficient 5/6 is consequently the model-dependent shape factor selected by this diagnostic, not a new universal constant; the more robust statement is that L_κ organizes the radial profile. The KMS relation in Eq. (69) fixes the thermal rate independently, while the atmosphere calculation asks where a Lorentzian mode ceases to evolve adiabatically. For a mode near the Planck-spectrum peak, $\omega_0 = \gamma/(2\pi L_\kappa)$ with $\gamma \simeq 2.82$, so the strength of the nonadiabaticity is likewise governed by the dimensionless thermal combination $\omega_0 L_\kappa = \gamma/(2\pi)$ [47]. The earlier tidal-work and renormalized-stress-tensor analysis found peaks at $r \simeq 4.32$–$4.38M \simeq (1.08$–$1.10)L_\kappa$ [46]. These independent diagnostics therefore place the proposed atmosphere in the band $r/L_\kappa \simeq 0.83$–1.10. The photon sphere and the peak of the eikonal Schwarzschild scattering potential, $r = 3M = 3L_\kappa/4$, lie in the same region, suggesting an interplay between the thermal/peeling scale and the optical potential. The two simple fractions have a common algebraic form without establishing a universal ladder: in the compactness variable $y = L_\kappa/(2r)$, the eikonal optical factor is proportional to $y^2(1-y)$ and peaks at $y = 2/3$, whereas the freely falling WKB diagnostic is proportional to $y^{3/2}(1-y)$ and peaks at $y = 3/5$. Their respective radii are $3L_\kappa/4$ and $5L_\kappa/6$ because distinct inward power laws are balanced against the same lapse factor. The stress-tensor and tidal peaks near $1.1L_\kappa$ are numerical, not exact rational ratios. Since Schwarzschild has only one macroscopic length and the detailed peaks depend on observer and diagnostic, this does not establish a universal emission shell or fraction sequence; it motivates testing whether atmosphere profiles collapse as functions of r/L_κ across fields and horizon families.

The same pressure has a familiar stationary field-stress correspondence. For a Newtonian gravitational field $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$, one conventional field-stress tensor is

$$t^{(g)}_{ij} = -\frac{1}{4\pi G}\left(g_i g_j - \frac{1}{2}\delta_{ij}|\boldsymbol{g}|^2\right). \tag{134}$$

For a surface normal parallel to $\boldsymbol{g}$, the longitudinal compressive stress magnitude is

$$P_g^{\parallel} \equiv -t^{(g)}_{nn} = \frac{|\boldsymbol{g}|^2}{8\pi G}. \tag{135}$$

At R_H, the Newtonian field magnitude is $|\boldsymbol{g}_N| = GM/R_H^2 = \kappa_\infty$, hence

$$P_g^{\parallel}(R_H) = P_L^{\mathrm{Schw}}. \tag{136}$$

This is a match of stationary, asymptotically normalized magnitudes. It does not turn P_L into a covariant local bulk stress. Rindler already shows that the boundary response is more general than a source-field interpretation [29–31].

E. What it means for the two halves to be distinct

The Schwarzschild Smarr identity can be written

$$\boxed{Mc^2 = TS + F_L L_\kappa, \qquad TS = F_L L_\kappa = \frac{Mc^2}{2}.} \tag{137}$$

This equality holds pointwise for every stationary Schwarzschild state. No time average is required. It is a gravitational scaling balance, expressed as complementary thermal and mechanical Euler products, and motivates a stationary virial interpretation. Static field-theory virial identities can follow from scaling an action without a time average; in general relativity the boundary terms are essential [61]. A direct test here is to scale the complete Einstein action in the same ensemble and identify its boundary balance with Eq. (137). That identification, and a separate kinetic/potential interpretation, are not proved by equality of the products alone.

For comparison, consider an ordinary bounded or periodic Hamiltonian system whose potential is homogeneous of degree n, $V(\lambda q) = \lambda^n V(q)$. The virial theorem gives

$$2\langle K\rangle = n\langle V\rangle, \qquad \langle K\rangle = \frac{n}{n+2}E, \qquad \langle V\rangle = \frac{2}{n+2}E, \tag{138}$$

when the displayed averages and denominator are well defined [32]. The theorem constrains the two averaged components across a family of total energies; it does not require one to decrease whenever the other increases. For $n > 0$, raising E raises both averages. A harmonic oscillator ($n = 2$) has equal halves. A complete ideal vertical projectile flight in uniform gravity, from launch at $z = 0$ to the instant it returns to $z = 0$ before any dissipative impact, has $n = 1$ and $\langle K\rangle = E/3$, $\langle V\rangle = 2E/3$: because the virial function $G = pz$ vanishes at both endpoints, the finite-time boundary term is exactly zero. Repeating the flight with elastic bounces makes the example periodic. The Kepler case $n = -1$ instead has negative binding energy, so adding positive energy makes the orbit less bound and changes the signs and magnitudes differently.

In ordinary mechanics, kinetic and potential energy are called distinct not because they are different substances, but because they are functions of different state-space data and generate different responses: kinetic energy depends on momentum or velocity and controls inertial motion, whereas potential energy depends on configuration and its gradient supplies force. They can be converted into one another while remaining operationally distinguishable energy forms.

The horizon split has an analogous, though not identical, structural basis. On the extended (A, L_κ) source space,

$$TS = A\left(\frac{\partial U_\partial}{\partial A}\right)_{L_\kappa}, \qquad F_L L_\kappa = -L_\kappa\left(\frac{\partial U_\partial}{\partial L_\kappa}\right)_A. \tag{139}$$

The first expression uses the area/entropy derivative and the second uses the normal-scale derivative. Both products are identically the same function $U_\partial(A, L_\kappa)$ even on the extended source space. The distinct differential legs are $\Pi_\partial\, dA$ and $-F_L\, dL_\kappa$. They do not establish independent kinetic and potential energies. Equation (26) also shows why these descriptions do not add a second scalar canonical degree of freedom.

There is an additional exact thermodynamic fact. The formal on-shell Schwarzschild Helmholtz free energy is

$$\boxed{\mathcal{F}_{\rm H}^{\rm on} = Mc^2 - TS = \frac{Mc^2}{2} = F_L L_\kappa.} \tag{140}$$

The scale product is precisely the mechanical/free-energy half of the stationary mass. Asymptotically flat Schwarzschild has negative heat capacity and is not a stable unconstrained canonical ensemble, but the on-shell Euclidean free-energy value remains well defined [33].

The thermal–mechanical interpretation therefore has a precise basis: TS is the thermal Euler product and $F_L L_\kappa$ the scale-work product, which also equals the formal free energy in this sector. Their balance is worth testing as a stationary virial relation. A literal kinetic/potential split would additionally require separately defined observables and a derivation of their dynamical or action-scaling roles.

Along the Schwarzschild curve,

$$d(TS) = d(F_L L_\kappa) = dU_\partial = F_L\, dL_\kappa = \frac{1}{2} d(Mc^2), \qquad T\, dS = d(Mc^2). \tag{141}$$

Adding black-hole mass increases the common value of the two Euler products by half the added mass energy. The boundary equation coexists exactly with the standard first law: the area leg contributes $2dU_\partial$ and the scale leg subtracts dU_∂, leaving the complete differential dU_∂.

F. Spherical horizon energy and the two length scales

The general potential is area based. In the spherical marginal-horizon specialization of Eq. (124), set

$$E_{\rm sph} = E_{\rm MS} = \frac{c^4 R}{2G}, \qquad A = 4\pi R^2. \tag{142}$$

The marginal condition is essential: $c^4 R/(2G)$ is not the Misner–Sharp energy of an arbitrary spherical cut. The previously established area potential becomes

$$\boxed{U_\partial = \frac{E_{\rm sph} R}{L_{\rm s}}, \qquad \frac{U_\partial}{E_{\rm sph}} = \frac{R}{L_{\rm s}} = \frac{\kappa R}{c^2}.} \tag{143}$$

Thus the ratio of the boundary potential to the geometric horizon energy is the ratio of transverse areal radius to normal acceleration length. On the positive branch $L_{\rm s} = L_\kappa$, Cai–Kim FRW has $L_\kappa = R_A$ and $U_\partial = E_A$; standard Schwarzschild has $L_\kappa = 2R_H$ and $U_\partial = Mc^2/2$. With a signed rate, Eq. (143) uses $L_{\rm s}$; magnitudes obey $|U_\partial|/E_{\rm sph} = R/L_\kappa$. The boundary potential retains its clock dependence even though $E_{\rm MS}$ is geometrically assigned.

The energy–length form unifies this interpretation with the established boost charge:

$$\boxed{U_\partial L_\mathrm{s} = cJ_\eta = E_\mathrm{sph} R.} \tag{144}$$

The first equality requires only the stated area–rate potential and Einstein boost normalization. The second uses a spherical marginal horizon. A Rindler patch therefore retains $U_\partial L_\mathrm{s} = cJ_\eta$ without any intrinsic spherical radius or enclosed Misner–Sharp energy. A formal area-equivalent radius $\sqrt{A/(4\pi)}$ labels size but does not create a curved sphere or its bulk energy.

Because $E_\mathrm{sph} \propto R$, its radius cannot be varied while holding that energy independently fixed. The complete differential is

$$\begin{aligned} dU_\partial &= \frac{R}{L_\mathrm{s}}\, dE_\mathrm{sph} + \frac{E_\mathrm{sph}}{L_\mathrm{s}}\, dR - \frac{E_\mathrm{sph} R}{L_\mathrm{s}^2}\, dL_\mathrm{s} \\ &= \frac{2U_\partial}{R}\, dR - F_L\, dL_\mathrm{s} = \Pi_\partial\, dA - F_L\, dL_\mathrm{s}. \end{aligned} \tag{145}$$

The two radius contributions together reproduce the area term; neither may be silently dropped. Section V G retains the additional changing-ratio term when L_s/R varies. The ER/L_s form is a useful spherical interpretation, not a broader replacement for $U_\partial(A, \kappa)$.

A useful distinction appears in the energy–temperature slopes. For $E = E_\mathrm{sph}$ on the positive branch, the ratio $\alpha = E/(TS)$ in the next formula equals $\alpha_R = L_\kappa/R$. For a spherical family with constant α, $E = \alpha TS$, $S \propto R^2$, and $T \propto R^{-1}$,

$$\mathcal{C}_E \equiv \frac{dE}{dT} = -\alpha S, \qquad C_\mathrm{path} \equiv T\frac{dS}{dT} = -2S. \tag{146}$$

The second coefficient is the reversible heat-path slope when $\delta Q_\mathrm{rev} = T\, dS$ applies. Schwarzschild has $\alpha = 2$ and the mass law $dE = T\, dS$, so its energy slope is the usual heat capacity $-2S$. Cai–Kim FRW has $\alpha = 1$: its energy slope is $-S$, while the screen-law heat-path slope is $-2S$, with their difference supplied by pressure work. These are not competing definitions of a single work-free heat capacity.

At fixed area in the extended source ensemble,

$$\chi_{UT}^{(A)} \equiv \left(\frac{\partial U_\partial}{\partial T}\right)_A = S, \qquad C_A^{(Q)} \equiv T\left(\frac{\partial S}{\partial T}\right)_A = 0. \tag{147}$$

The first is an energy–temperature susceptibility. Entropy is fixed on that slice, so the variation changes the source potential through the work channel rather than transferring reversible heat.

G. Radius force, homogeneity, and the limits of the Smarr connection

Equations (S27)–(S29) follow by substituting $A = 4\pi R^2$ and $L_\mathrm{s} = \alpha_R R$ into the parent potential:

$$U_\partial(R, \alpha_R) = \frac{c^4 R}{2G\alpha_R}, \qquad \Pi_\partial\, dA = 2F_R\, dR, \qquad F_L\, dL_\mathrm{s} = F_R\, dR + U_\partial \frac{d\alpha_R}{\alpha_R}. \tag{148}$$

The positive radius-work pressure is $P_R = F_R/A$ on the positive branch. It differs from the acceleration-length pressure by $P_R = \alpha_R P_L$. The identity

$$U_\partial = F_R R = 3P_R V = 3\alpha_R P_L V \tag{149}$$

uses the Euclidean areal-volume convention $AR = 3V$. It does not make $A\, dL_\mathrm{s}$ a general exact volume differential and does not identify P_R with a matter pressure or a cosmological-constant pressure.

Under a physical rescaling with dimensionless sector data fixed, $R \mapsto \lambda R$, $A \mapsto \lambda^2 A$, $L_\mathrm{s} \mapsto \lambda L_\mathrm{s}$, and $\kappa \mapsto \kappa/\lambda$. The boundary potential has degree one:

$$U_\partial(\lambda^2 A, \lambda L_\mathrm{s}) = \lambda U_\partial(A, L_\mathrm{s}), \qquad 2\Pi_\partial A - F_L L_\mathrm{s} = U_\partial. \tag{150}$$

This is the same Euler identity as $R(\partial U_\partial/\partial R)_{\alpha_R} = U_\partial$. The radius representation displays the homogeneity underlying Smarr relations [56, 57]; the acceleration-length representation already contains it. No additional Smarr constraint follows merely from the change of variables.

For Schwarzschild, the mass first law and its integrated relation are

$$d(Mc^2) = \Pi_\partial\, dA = 2M\kappa\, dR_H, \qquad Mc^2 = 2U_\partial = 2M\kappa R_H = TS + M\kappa R_H. \tag{151}$$

In particular, $M\kappa\, dR_H = dU_\partial = \frac{1}{2}d(Mc^2)$. The two equal products TS and $M\kappa R_H$ are representations of one function U_∂, not independent energy reservoirs. For Cai–Kim FRW, by contrast, $\alpha_R = 1$, $U_\partial = E_A = M_A c^2$, and

$$F_R = M_A \kappa_{\rm CK} = \frac{c^4}{2G}, \qquad E_A = F_R R_A = 3P_{\rm scr} V_A, \qquad dE_A = 2F_R\, dR_A - F_R\, dR_A. \tag{152}$$

This is the analogous horizon-energy scaling identity within the Cai–Kim construction; it does not supply an asymptotic FRW mass or a stationary black-hole mass theorem.

The condition on the scale ratio is essential. For signed Hayward FRW, away from $\kappa_H^{\rm s} = 0$, the established rate relation gives $\alpha_R = 4/(3w-1)$. The second term of Eq. (148) is then $U_\partial\, d\alpha_R/\alpha_R = -3E_A\, dw/4$, exactly the equation-of-state contribution derived in Sec. VI. Only at fixed w does this work reduce to $(M_A \kappa_H^{\rm s})\, dR_A$.

a. Kerr: an exact mass–radius product with a specified radius. For an asymptotically flat, nonextremal, uncharged Kerr black hole with the standard infinity-normalized stationary generator, let r_+ be the outer Boyer–Lindquist horizon coordinate and let $a_{\rm K} = J/(Mc)$. The established Kerr relations [55–57] imply

$$A = 4\pi(r_+^2 + a_{\rm K}^2) = \frac{8\pi G M r_+}{c^2}, \qquad \boxed{U_\partial = M\kappa r_+.} \tag{153}$$

Defining $\alpha_+ = L_{\rm s}/r_+$ therefore gives the exact scale-work identity

$$F_L\, dL_{\rm s} = M\kappa\, dr_+ + U_\partial \frac{d\alpha_+}{\alpha_+}, \qquad Mc^2 = 2M\kappa r_+ + 2\Omega_H J. \tag{154}$$

For $\chi = Jc/(GM^2)$ and $s = \sqrt{1-\chi^2} > 0$, $\kappa r_+/c^2 = s/2$ and $\alpha_+ = 2/s$. Hence $F_L\, dL_{\rm s} = M\kappa\, dr_+$ along fixed-dimensionless-spin scaling families. Fixed J or fixed $a_{\rm K}$ is generally a different variation and retains the extra term. The standard mass first law remains $d(Mc^2) = \Pi_\partial\, dA + \Omega_H\, dJ$.

The coordinate r_+ is neither an areal radius nor a proper radial distance. If instead $R = \sqrt{A/(4\pi)}$ is used, the general radius product is $U_\partial = M_{\rm irr}\kappa R$, where $M_{\rm irr} = c^2 R/(2G)$ is the irreducible mass, not the Kerr total mass. The spherical canonical normalization in Eq. (36) must not be transferred unchanged to $(r_+, M\kappa)$ when spin and horizon shape also vary.

b. Charged horizons and absence of a preferred radius. A simple counterexample to an unrestricted total-mass force claim is Reissner–Nordström. In $G = c = 1$ units, with outer areal radius R and geometrized charge Q,

$$M = \frac{R^2 + Q^2}{2R}, \quad M_R = \frac{R}{2}, \quad \kappa = \frac{1 - Q^2/R^2}{2R}, \quad -\frac{A}{8\pi}\left(\frac{\partial \kappa}{\partial R}\right)_Q = \frac{1}{4}\left(1 - \frac{3Q^2}{R^2}\right). \tag{155}$$

The last coefficient is the scale-work coefficient along a fixed-Q radius variation. It is generally neither $M_R\kappa$ nor $M\kappa$; the scale-ratio term accounts for the difference. At fixed Q/R, α_R is constant and the coefficient is $M_R\kappa$, while M_R still differs from the total mass. Thus both the mass definition and the variation must be specified. Rindler supplies a different limit: its finite-patch boundary potential is well defined, but there is no preferred spherical radius or enclosed Misner–Sharp mass. Acceleration length remains meaningful there.

The two representations have complementary uses. Radius makes the spherical scaling and the stated Kerr Smarr product especially direct; acceleration length retains the operational Rindler normalization and the normal-rate source interpretation before a radius prescription is chosen. Neither produces an additional law merely by reparameterizing the same potential.

H. Schwarzschild–de Sitter and clock normalization

Schwarzschild hides the normalization issue because asymptotic flatness supplies a conventional clock. For

$$f(r) = 1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3} \tag{156}$$

and a static generator $\chi_\alpha^a = \alpha(\partial_t)^a$, either horizon r_i has

$$|\kappa_i^{(\alpha)}| = \alpha \frac{c^2}{2r_i}|1 - \Lambda r_i^2|, \qquad P_{L,i}^{(\alpha)} = \frac{[\kappa_i^{(\alpha)}]^2}{8\pi G}. \tag{157}$$

Different admissible normalizations assign different numerical responses to the same intrinsic cut. Once a particular static observer is selected, the locally redshifted value is consistent. The absence of a preferred global clock is therefore a physical ensemble issue, not an algebraic inconsistency [48, 49].

VI. COSMOLOGICAL APPLICATION: FRW SCREEN WORK AND NORMAL FOCUSING

A. Compact-screen response

Consider the four-dimensional FRW metric

$$ds^2 = -c^2dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2\right), \qquad R(t,r) = a(t)r, \tag{158}$$

and let ρ and $p_{\rm m}$ denote total energy density and matter pressure in the same units. The apparent-horizon radius is

$$R_A = \frac{c}{\sqrt{H^2 + kc^2/a^2}}. \tag{159}$$

The Friedmann constraint and Misner–Sharp definition give

$$E_A = \rho V_A = \frac{c^4 R_A}{2G}, \qquad V_A = \frac{4\pi}{3}R_A^3, \qquad \rho = \frac{3c^4}{8\pi G R_A^2}. \tag{160}$$

Here V_A is the areal volume entering the Misner–Sharp relation. For $k \neq 0$ it differs from the proper volume of a cosmic-time spatial ball; the latter contains the radial metric factor $1/\sqrt{1-kr^2}$. Along the homogeneous family of apparent-horizon screens,

$$\boxed{P_{\rm scr} \equiv \frac{dE_A}{dV_A} = \frac{c^4}{8\pi G R_A^2} = \frac{\rho}{3}.} \tag{161}$$

This is a positive compact-screen size response, not the cosmic-fluid pressure $p_{\rm m}$ and not the conventional independent-variable derivative $-(\partial E/\partial V)_S$. Both S_A and V_A are functions of the single physical parameter R_A.

The associated force is

$$F_{\rm scr} = A_A P_{\rm scr} = \frac{c^4}{2G}, \qquad dE_A = F_{\rm scr}\, dR_A = P_{\rm scr}\, dV_A. \tag{162}$$

The force is again constant in four dimensions, now four times the Schwarzschild value because the Cai–Kim acceleration scale is twice the Schwarzschild surface gravity at the same radius.

The inverse pressure slope along the physical spherical family is

$$\boxed{\frac{dV_A}{dP_{\rm scr}} = -\frac{3V_A}{2P_{\rm scr}}, \qquad B_{\rm scr} \equiv -V_A\frac{dP_{\rm scr}}{dV_A} = \frac{2}{3}P_{\rm scr}.} \tag{163}$$

The factor of three relative to Eq. (107) arises because the transverse area changes with the radius rather than being held fixed. This is a constrained-screen response slope, not a dynamical stability result. Kong's effective pressure $P_{\rm eff} \equiv -dE_A/dV_A$ is consequently $-P_{\rm scr} = -\rho/3$ in four-dimensional Einstein FRW [41]; it uses a different work convention and is not the positive scale-work coefficient in Eq. (177).

B. Friedmann system in screen-pressure variables

Define the Einstein coupling in the field-equation normalization,

$$\kappa_{\rm E} \equiv \frac{8\pi G}{c^4}. \tag{164}$$

Since $\rho = 3P_{\rm scr}$, the two Friedmann equations and the continuity equation can be written entirely in terms of the positive screen pressure $P_{\rm scr}$ and the signed matter pressure $p_{\rm m}$:

$$\frac{1}{R_A^2} = \kappa_{\rm E} P_{\rm scr}, \tag{165}$$

$$\frac{2}{c^2}\left(\dot{H} - \frac{kc^2}{a^2}\right) = -\kappa_{\rm E}(3P_{\rm scr} + p_{\rm m}), \tag{166}$$

$$\frac{2\ddot{a}}{ac^2} = -\kappa_{\rm E}(P_{\rm scr} + p_{\rm m}), \tag{167}$$

$$\dot{P}_{\rm scr} + H(3P_{\rm scr} + p_{\rm m}) = 0. \tag{168}$$

Equation (165) says that the screen pressure sets the apparent-horizon inverse-square scale. The acceleration equation shows that the universe accelerates precisely when the negative bulk pressure exceeds the positive screen pressure in magnitude, $p_{\rm m} < -P_{\rm scr}$. The combination $3P_{\rm scr} + p_{\rm m}$ governs the change of the Hubble rate and the dilution of the screen pressure. Equivalently,

$$\boxed{\frac{\dot{R}_A}{HR_A} = \frac{3P_{\rm scr} + p_{\rm m}}{2P_{\rm scr}}.} \tag{169}$$

The geometry clarifies why these combinations appear. Spherical spacetime separates as

$$ds^2 = h_{ab}dx^a dx^b + R^2(x)d\Omega_2^2, \tag{170}$$

where h_{ab} is the two-dimensional Lorentzian time–radial, or orbit, geometry and the second term is the transverse symmetry sphere. On the apparent-horizon cut, the sphere has Gaussian curvature

$$\boxed{K_{S^2} = \frac{1}{R_A^2} = \kappa_{\rm E} P_{\rm scr}.} \tag{171}$$

This equality is special to the Cai–Kim realization $L_{\rm CK} = R_A$; the general boundary-pressure identity is $\kappa_{\rm E} P_L = 1/L_\kappa^2$. The scalar curvature of the normal time–radial geometry is, with the curvature convention used here,

$$\boxed{{}^{(2)}\mathcal{R}_\perp = \frac{2\ddot{a}}{ac^2} = -\kappa_{\rm E}(P_{\rm scr} + p_{\rm m}).} \tag{172}$$

The transverse two-sphere Ricci scalar is ${}^{(2)}\mathcal{R}_{S^2} = 2/R_A^2 = 2\kappa_{\rm E}P_{\rm scr}$. Their sum isolates the trace-sensitive pressure contrast,

$$\boxed{{}^{(2)}\mathcal{R}_{S^2} + {}^{(2)}\mathcal{R}_\perp = \kappa_{\rm E}(P_{\rm scr} - p_{\rm m}) = \frac{{}^{(4)}\mathcal{R}}{3}.} \tag{173}$$

Therefore $P_{\rm scr}$ fixes transverse horizon curvature, $P_{\rm scr} + p_{\rm m}$ fixes normal expansion curvature, and $P_{\rm scr} - p_{\rm m}$ fixes the trace or cross-focusing curvature.

C. Cai–Kim equilibrium scale

A generic FRW apparent-horizon worldtube is not a non-expanding null boundary: a marginal cut does not imply a null, non-expanding tube.

A marginal cut has zero expansion along one null normal, whereas the worldtube follows the locus of marginal cuts and need not follow that normal. For flat dust FRW, $R_A = 3ct/2$ and its speed relative to the comoving frame is $\dot{R}_A - HR_A = c/2$: the tube is timelike and its area changes. A stationary de Sitter horizon is a special nonexpanding null overlap. Thus stationary Rindler, Schwarzschild, and de Sitter admit the direct restricted certification, while general CK FRW is the separate screen-state realization stated here. The null-source proposition therefore does not by itself establish a canonical theorem for this evolving screen. The following realization uses the Friedmann constraint and the separately specified instantaneous Cai–Kim calibration,

$$\kappa_{\rm CK} = \frac{c^2}{R_A}, \qquad L_{\rm CK} = R_A, \qquad T_{\rm CK} = \frac{\hbar c}{2\pi k_B R_A}. \tag{174}$$

It makes the acceleration-length pressure coincide exactly with the screen response:

$$P_L^{\rm CK} = P_{\rm scr}, \qquad T_{\rm CK} S_A = E_A, \qquad F_L^{\rm CK} = M_A \kappa_{\rm CK} = \frac{c^4}{2G}, \tag{175}$$

where $M_A = E_A/c^2$. Because $L_{\rm CK} = R_A$, the same force is conjugate to both length and radius, and $F_L^{\rm CK} dL_{\rm CK} = M_A \kappa_{\rm CK} dR_A = P_{\rm scr} dV_A$. The differential chain is

$$T_{\rm CK} dS_A = 2dE_A, \qquad -S_A dT_{\rm CK} = F_L^{\rm CK} dL_{\rm CK} = P_{\rm scr} dV_A = dE_A, \tag{176}$$

and hence

$$\boxed{dE_A = T_{\rm CK} dS_A - P_{\rm scr} dV_A.} \tag{177}$$

This is the cleanest literal $P\,dV$ realization of the normal scale response. Every quantity is fixed by the same macroscopic radius. The result is an exact reorganization of the Einstein–FRW constraint, not a new Friedmann equation and not Cai and Kim's original matter-flux identity [37].

The distinction matters. Cai and Kim used the energy supply crossing a momentarily fixed apparent horizon,

$$\delta Q_{\rm matt} = A_A(\rho + p_{\rm m}) H R_A\, dt = T_{\rm CK} dS_A, \tag{178}$$

with signs determined by orientation. Equation (177) instead resolves the simultaneous change of the constrained screen state into entropy and size legs. The flux law sorts change by carrier; the screen law sorts the same geometry by boundary coordinate.

The energy–temperature slope from Eq. (146) is

$$\frac{dE_A}{dT_{\rm CK}} = -S_A, \qquad T_{\rm CK}\frac{dS_A}{dT_{\rm CK}} = -2S_A, \qquad -P_{\rm scr}\frac{dV_A}{dT_{\rm CK}} = S_A. \tag{179}$$

The work term accounts for the difference. The second expression is the heat-path coefficient of the formal screen-state first law, not a claim of exact equilibrium thermality for arbitrary evolving FRW geometry.

D. Full Hayward surface gravity

The Cai–Kim scale is an instantaneous equilibrium calibration. The signed Kodama–Hayward surface gravity probes the evolving normal two-geometry:

$$\kappa_H^{\rm s} = \frac{c^2}{2}\Box_h R\big|_{R_A} = -\frac{c^2}{R_A}\left(1 - \frac{\dot R_A}{2HR_A}\right). \tag{180}$$

The ratio form assumes $H \neq 0$. A regular expression at $H = 0$, when R_A remains finite, is $\kappa_H^{\rm s} = -(R_A/2)(\dot H + 2H^2 + kc^2/a^2)$. It measures the transverse-null slope of the marginal condition, or normal cross-focusing. A comoving FRW observer is geodesic, so this is not the observer's proper acceleration.

Using the Friedmann and continuity equations,

$$\frac{\dot R_A}{2HR_A} = \frac{3P_{\rm scr} + p_{\rm m}}{4P_{\rm scr}} = \frac{3}{4}(1 + w), \qquad w \equiv \frac{p_{\rm m}}{\rho}, \tag{181}$$

for arbitrary spatial curvature. Therefore the signed Hayward surface gravity admits the equivalent pressure and curvature forms

$$\boxed{\kappa_H^{\rm s} = \frac{c^2}{4R_A}(3w - 1) = -\frac{c^2 R_A}{4}\kappa_{\rm E}(P_{\rm scr} - p_{\rm m}) = -\frac{c^2 R_A}{12}{}^{(4)}\mathcal{R}.} \tag{182}$$

It measures the trace-sensitive contrast between the positive screen pressure and the signed material pressure.

a. Ricci-trace surface density: exact content and status. The dependence of the FRW Kodama–Hayward rate and temperature on $\rho - 3p_{\rm m}$, together with the radiation threshold $p_{\rm m} = \rho/3$, is established in the apparent-horizon literature [40]. The additional step here is to package that trace combination as a signed surface density. Define

$$E_{\rm tr} \equiv (\rho - 3p_{\rm m})V_A, \qquad M_{\rm tr} \equiv \frac{E_{\rm tr}}{c^2}, \qquad \Sigma_{\rm tr} \equiv \frac{M_{\rm tr}}{A_A}. \tag{183}$$

$E_{\rm tr}$ is an integrated signed stress–energy trace; it is not generally a conserved material energy or a positive rest mass. Using $V_A/A_A = R_A/3$ and $\kappa_{\rm CK} = c^2/R_A$ gives

$$\boxed{\Sigma_{\rm tr} = \frac{(\rho - 3p_{\rm m})R_A}{3c^2} = \frac{P_{\rm scr} - p_{\rm m}}{\kappa_{\rm CK}}, \qquad \kappa_H^{\rm s} = -2\pi G\Sigma_{\rm tr} = -\frac{GM_{\rm tr}}{2R_A^2}.} \tag{184}$$

The exact statement is that the signed Hayward cross-focusing rate is determined by the bulk Ricci-trace mass equivalent per horizon area. Where a Hayward temperature is meaningful,

$$\boxed{k_B T_H = \frac{\hbar|\kappa_H^{\rm s}|}{2\pi c} = \frac{\hbar G}{c}|\Sigma_{\rm tr}|.} \tag{185}$$

Faraoni's temperature formula already contains this result in the unpackaged form $R_A(\rho - 3p_{\rm m})$; Eq. (184) is therefore a compact boundary-density representation rather than a new dynamical equation.

A complementary active surface density is

$$\boxed{\Sigma_{\rm act} \equiv \frac{(\rho + 3p_{\rm m})V_A}{c^2 A_A} = \frac{P_{\rm scr} + p_{\rm m}}{\kappa_{\rm CK}}, \qquad \frac{\ddot{a}}{a}R_A = -4\pi G\Sigma_{\rm act}.} \tag{186}$$

The pair separates two familiar projections of Einstein's equation: $\rho + 3p_{\rm m}$ controls timelike relative acceleration, whereas $\rho - 3p_{\rm m}$ controls the Ricci trace and FRW Hayward cross-focusing. The third combination $\rho + p_{\rm m} = 3P_{\rm scr} + p_{\rm m}$ controls null energy flux and the evolution of R_A.

The broader four-dimensional spherical Hayward identity in Eq. (218) can itself be written as

$$\boxed{\kappa_H^{\rm s} = 4\pi G(\Sigma_E - \Sigma_W), \qquad \Sigma_E \equiv \frac{E}{Ac^2}, \qquad \Sigma_W \equiv \frac{RW}{c^2}.} \tag{187}$$

FRW homogeneity gives $\Sigma_E - \Sigma_W = -\Sigma_{\rm tr}/2$. The trace form is not the general spherical law: in Vaidya null dust $W = 0$ and the matter trace vanishes while $\kappa_H = GE/(c^2R^2)$ remains nonzero; in inhomogeneous dust the first term contains the average Misner–Sharp density while the work term contains local matter data. Homogeneity is what turns the general surface-density balance into the simple Ricci-trace density.

The coefficient $2\pi G$ is the same numerical conversion that gives the one-sided Newtonian field of an ideal plane, or the mean self-field of a thin spherical shell. No distributional shell is present in smooth FRW, so this is a mass-sheet form rather than a demonstrated mass sheet.

On a nonzero branch, the corresponding signed Hayward acceleration length is

$$\boxed{L_H^{\rm s} \equiv \frac{c^2}{\kappa_H^{\rm s}} = \frac{4R_A P_{\rm scr}}{p_{\rm m} - P_{\rm scr}} = \frac{4R_A}{3w - 1}, \qquad \frac{R_A}{L_H^{\rm s}} = \frac{p_{\rm m} - P_{\rm scr}}{4P_{\rm scr}}.} \tag{188}$$

The quadratic Hayward response becomes

$$\boxed{P_L^H = \frac{(\kappa_H^{\rm s})^2}{8\pi G} = \frac{(P_{\rm scr} - p_{\rm m})^2}{16P_{\rm scr}} = \left(\frac{1 - 3w}{4}\right)^2 P_{\rm scr}.} \tag{189}$$

P_L^H and $P_{\rm scr}$ answer different questions. For dust, $P_L^H = P_{\rm scr}/16$ and $|L_H^{\rm s}| = 4R_A$; for radiation the pressure contrast vanishes, so $P_L^H = 0$ and $|L_H^{\rm s}| = \infty$; at the coasting threshold $p_{\rm m} = -P_{\rm scr}$, $|L_H^{\rm s}| = 2R_A$; and for de Sitter $P_L^H = P_{\rm scr}$ and $|L_H^{\rm s}| = R_A$.

b. Quintessence in screen-pressure variables. Because $P_{\rm scr} = \rho/3$, accelerated expansion is the pressure inequality $p_{\rm m} < -P_{\rm scr}$. If a canonical scalar field dominates, let K_ϕ and V_ϕ denote its kinetic and potential energy densities, so $\rho_\phi = K_\phi + V_\phi$ and $p_\phi = K_\phi - V_\phi$; the acceleration condition becomes $V_\phi > 2K_\phi$. The canonical accelerating-quintessence range $-1 < w_\phi < -1/3$ is equivalently

$$\boxed{-3P_{\rm scr} < p_\phi < -P_{\rm scr}, \qquad R_A < |L_H^{\rm s}| < 2R_A, \qquad \frac{1}{4}P_{\rm scr} < P_L^H < P_{\rm scr}.} \tag{190}$$

The cosmological-constant limit has $|L_H^{\rm s}| = R_A$ and $P_L^H = P_{\rm scr}$, while the coasting limit has $|L_H^{\rm s}| = 2R_A$ and $P_L^H = P_{\rm scr}/4$.

Using $V_A = A_A R_A/3$, Eq. (188) also has the purely spherical form

$$L_H^{\rm s} = \frac{4V_A/A_A}{w - 1/3} = \frac{V_A}{(A_A/4)(w - 1/3)}. \tag{191}$$

Here $4V_A/A_A = 4R_A/3$ is the sphere's volume-to-area length. The appearance of $A_A/4 = \pi R_A^2$ is ordinary spherical geometry; no common origin with the Bekenstein–Hawking quarter-area coefficient is implied.

Faraoni showed that the expanding apparent horizon changes trapping and oriented-temperature character at

$$p_{\rm m} = \frac{\rho}{3} = P_{\rm scr}. \tag{192}$$

The threshold is established work; the additional reading here is that it equates matter pressure with the independently defined compact-screen response. At the threshold, $\kappa_H^{\rm s} = 0$ and $P_L^H = 0$ while $P_{\rm scr}$ remains finite. Geometrically, the positive transverse-sphere curvature and negative normal-plane curvature cancel in Eq. (173). The rate variable remains regular, but its reciprocal length chart reaches infinity [40].

A generic FRW apparent horizon is a quasi-local marginal screen rather than a cosmological event horizon. Its worldtube can be timelike, spacelike, or null, and it can be crossed as the cosmology evolves. Exact de Sitter is the special case in which the apparent and event horizons coincide.

E. Hayward–Mukohyama–Ashworth parent potential

Hayward's work density is

$$W = -\frac{1}{2}T^{ab}h_{ab} = \frac{\rho - p_{\rm m}}{2}. \tag{193}$$

The spherical Einstein equations and the Hayward–Mukohyama–Ashworth Kodama–Noether charge give

$$\widetilde{U_\partial} \equiv \frac{A_A c^2 \kappa_H^{\rm s}}{8\pi G} = \frac{E_A}{2} - \frac{3}{2}WV_A = -\frac{1}{4}(\rho - 3p_{\rm m})V_A = -\frac{1}{4}E_{\rm tr}. \tag{194}$$

The HMA parent potential is therefore minus one quarter of the signed trace energy. Combining this identity with $\widetilde{U_\partial} = A_A c^2 \kappa_H^{\rm s}/(8\pi G)$ reproduces Eq. (184) directly. This supplies an independent quasi-local route to the surface-density packaging: the Hayward rate is controlled by Ricci-trace energy per horizon area, while the Cai–Kim screen pressure remains controlled by total energy per screen volume.

On a branch admitting a positive Hayward temperature,

$$\boxed{T_H S_A = |\widetilde{U_\partial}| = \frac{|E_{\rm tr}|}{4}.} \tag{195}$$

The quarter in Eq. (195) and the quarter-area entropy coefficient share the same $A\kappa/(8\pi G)$ Noether normalization and a 2π conversion. They are distinct physical statements, not a microscopic derivation of one from the other.

Pulling a variation back along the physical trapping-horizon trajectory, Hayward's unified first law is

$$d_H E_A = T_H^{\rm s} d_H S_A + W\, d_H V_A. \tag{196}$$

For nonzero $\kappa_H^{\rm s}$, differentiation of Eq. (194) gives the exact allocation

$$\boxed{-F_L^H d_H L_H^{\rm s} = -\frac{1}{2}T_H^{\rm s} d_H S_A - W\, d_H V_A - \frac{3}{2}V_A d_H W.} \tag{197}$$

This shows precisely how the reciprocal-rate variation is redistributed among area, matter-work, and changing-work-density terms. The Hayward acceleration length therefore supplies a normal-scale interpretation of the established dynamical balance rather than an additional standalone first law. It is the appropriate dynamical qualification to the equilibrium Cai–Kim law [19, 38].

F. Exact relation between the two cosmological scale-work terms

The main pressure variables are $P_{\rm scr} = \rho/3$ and the material pressure $p_{\rm m}$. For $\rho > 0$, the standard equation-of-state parameter is $w = p_{\rm m}/\rho = p_{\rm m}/(3P_{\rm scr})$. The dimensionless ratio

$$\chi \equiv \frac{\kappa_H^{\rm s}}{\kappa_{\rm CK}} = -\frac{P_{\rm scr} - p_{\rm m}}{4P_{\rm scr}} = \frac{3w - 1}{4} \tag{198}$$

is a useful algebraic shorthand; it introduces no new physical source. The dynamical rate depends on the pressure contrast, whereas cosmic acceleration depends on the pressure sum. Explicitly,

$$\boxed{P_L^H = \frac{(P_{\rm scr}-p_{\rm m})^2}{16P_{\rm scr}}, \qquad \widetilde{U_\partial} = -\frac{3}{4}(P_{\rm scr}-p_{\rm m})V_A.} \tag{199}$$

On a branch $\chi \neq 0$, the HMA and Cai–Kim potentials obey

$$\widetilde{U_\partial} = \chi E_A, \qquad L_H^{\rm s} = \frac{R_A}{\chi}, \qquad P_L^H = \chi^2 P_{\rm scr}, \qquad F_L^H = \chi^2 F_L^{\rm CK}. \tag{200}$$

Consequently the exact scale-work transformation is

$$\boxed{F_L^H dL_H^{\rm s} = \chi P_{\rm scr} dV_A - E_A d\chi = \chi P_{\rm scr} dV_A - \frac{3}{4} E_A dw.} \tag{201}$$

In the pressure notation used for the Friedmann equations this is

$$\boxed{F_L^H dL_H^{\rm s} = \frac{p_{\rm m}-P_{\rm scr}}{4}\, dV_A - \frac{3}{4} E_A\, dw.} \tag{202}$$

The familiar w is retained only to make the changing-equation-of-state contribution transparent. Equivalently, its last term is $-E_A\, d(p_{\rm m}/P_{\rm scr})/4$; no new equation-of-state variable is needed. At fixed w the work reduces to $F_L^H dL_H^{\rm s} = M_A \kappa_H^{\rm s} dR_A$. On a positive-temperature branch use $f = |\chi|$, $L_H = R_A/f$, and $U_H^+ = fE_A$; then $F_L^H dL_H = fP_{\rm scr} dV_A - E_A df$, with radial force $M_A|\kappa_H^{\rm s}|$ at fixed w.

The two energy balances are simultaneously consistent:

$$\boxed{dE_A = \Pi_{\rm CK} dA_A - P_{\rm scr} dV_A = \Pi_H^{\rm s} dA_A + W dV_A.} \tag{203}$$

Using $\Pi_{\rm CK} = R_A P_{\rm scr}$, $\Pi_H^{\rm s} = R_A(p_{\rm m}-P_{\rm scr})/4$, and $W = (3P_{\rm scr}-p_{\rm m})/2$, the same pair reads

$$\boxed{\begin{aligned} dE_A &= R_A P_{\rm scr}\, dA_A - P_{\rm scr}\, dV_A \\ &= \frac{R_A(p_{\rm m}-P_{\rm scr})}{4}\, dA_A + \frac{3P_{\rm scr}-p_{\rm m}}{2}\, dV_A. \end{aligned}} \tag{204}$$

These are balances for E_A, not differentials of the signed HMA potential $\widetilde{U_\partial}$. The second equality is pulled back to the physical trapping-horizon trajectory. Since $dA_A = 2dV_A/R_A$ and $2(1-\chi)P_{\rm scr} = P_{\rm scr} + W$, the different area coefficients are compensated by their different work allocations. This is an exact Einstein–FRW relation, independent of spatial curvature. It does not identify either rate with the complete source of every moving-boundary action; Appendix C gives a separate comparison that leaves this general matching problem unresolved. At $\chi = 0$, use the regular rate and potential variables instead of the singular reciprocal chart.

G. Operational audit of the trace-surface-density interpretation

a. Published membrane and Brown–York tests. A direct FLRW membrane construction by Wang uses a generator for which $g_H = H$ and, in units $c = 1$, obtains at the apparent horizon

$$\rho_H = 0, \qquad \widetilde{p}_H = \frac{H}{8\pi G}. \tag{205}$$

The result is an oblique, generator-dependent membrane, and its surface energy density is not $\Sigma_{\rm tr}$ [50]. The Brown–York analysis of Oltean, Bazrafshan Moghaddam, and Epp likewise gives for rigid FLRW spheres a quasilocal energy density proportional to γ^{-1}, which tends to zero as the rigid worldtube approaches the apparent horizon [51]. If the flat-space vacuum term is subtracted, the limiting mass per area is instead $c^2/(4\pi G R_A)$, independent of w, whereas

$$\Sigma_{\rm tr} = (1-3w)\frac{c^2}{8\pi G R_A}. \tag{206}$$

The two agree only accidentally at $w = -1/3$. These established quasilocal constructions therefore do not identify $\Sigma_{\rm tr}$ as the physical membrane energy density.

b. Junction test. A literal infinitesimal shell requires a discontinuity of first metric derivatives and a delta-function surface stress. For stationary null horizons, the corresponding surface stress is controlled by jumps such as $[\kappa]$ [52]. A smooth FRW apparent horizon has no such jump. Equation (184) therefore describes a smooth quasi-local rate, not an Israel or Barrabès–Israel shell.

c. Normal-frame boost test. The Hayward rate is also not the time derivative of the naive rapidity of the apparent-horizon worldtube. In spatially flat dust FRW, $a(t) \propto t^{2/3}$ and $R_A = 3ct/2$. The horizon moves relative to the comoving flow with constant speed $\dot{R}_A - HR_A = c/2$, so that rapidity is constant, while $\kappa_H^{\mathrm{s}} = -c^2/(4R_A) \neq 0$. Any corner interpretation must therefore use the Kodama-adapted normal-bundle connection and moving-cut terms, not the elementary worldtube velocity.

d. Dimensional test. For n spatial dimensions, the Einstein–FRW equations give

$$\frac{\dot{R}_A}{2HR_A} = \frac{n}{4}(1+w), \qquad \frac{\kappa_H^{\mathrm{s}} R_A}{c^2} = \frac{n(1+w)-4}{4}. \tag{207}$$

This is proportional to the stress trace $\rho - np$ only for $n = 3$. The simple $\rho - 3p$, $-E_{\mathrm{tr}}/4$, and $-2\pi G\Sigma_{\mathrm{tr}}$ relations are therefore specifically four-dimensional.

e. What a broader canonical theorem would mean. Two extensions are different: a null hypersurface with nonzero expansion and shear, and a marginal-horizon worldtube that can be timelike, spacelike, or null. Equation (29) controls one scalar part of the first problem, not the complete second problem. A full result for a specified cosmological rate would derive the on-shell variation with the actual embedding, matter, clock, stretching, and corner data, then identify the coefficient of the original intended L_{CK} or L_H^{s} while retaining the other terms. The explicit mismatch in Appendix C excludes that identification for its chosen ensemble, not for every possible ensemble. The missing matching step is not completed by renaming its natural source as surface gravity.

f. Scope of the moving-boundary comparison. A Kodama-adapted Carrollian polarization remains a useful test of the HMA stress $\Pi_H = c^2\kappa_H^{\mathrm{s}}/(8\pi G) = -c^2\Sigma_{\mathrm{tr}}/4$ [53, 60]. It must retain clock and stretching responses, not only the pressure–area pair. Appendix C examines an explicit timelike area-mixed action. Its rescaled surface stress differs from κ_H^{s} on a generic evolving screen, so that calculation does not complete a Hayward or Cai–Kim canonical extension. Tracking, peeling, and Kodama–Hayward rates must be compared within specified prescriptions [54].

H. Inflationary tensor spectrum in screen-pressure variables

The equilibrium screen pressure also gives a mechanical representation of a standard quantum observable. Consider spatially flat, quasi-de Sitter inflation in four-dimensional Einstein gravity, with constant G, the usual Bunch–Davies vacuum, and linear tensor perturbations without additional tensor sources. The tensor action and its quantization are extra inputs; they do not follow from the boundary state law. We use the instantaneous Cai–Kim assignment $L_\kappa = R_A = c/H$ and $\kappa_{\mathrm{CK}} = cH$, not the full dynamical Hayward rate. Denote a comoving wave number by q to distinguish it from the spatial-curvature parameter k. A star denotes evaluation at its Hubble-scale exit, $cq = a_* H_*$.

a. Amplitude, pressure, and entropy. The dimensionless primordial tensor power $\Delta_t^2(q)$ is the variance per logarithmic wave-number interval, summed over the two polarizations. It is not a pressure, radiated power in watts, or the energy density of gravitational waves. The conventional leading slow-roll result is [69]

$$\Delta_t^2(q) \simeq \frac{16\hbar G H_*^2}{\pi c^5}. \tag{208}$$

The same background has

$$P_{\mathrm{scr}} = \frac{c^2H^2}{8\pi G}, \qquad N_S \equiv \frac{S_A}{k_B} = \frac{A_A}{4\ell_{\mathrm{P}}^2} = \frac{\pi c^5}{\hbar G H^2}, \qquad P_{\mathrm{Pl}} \equiv \frac{c^7}{\hbar G^2}. \tag{209}$$

Here P_{Pl} is a defined Planck pressure, not an assumed upper bound. The pressure–entropy product and tensor representation are

$$P_{\mathrm{scr}} N_S = \frac{P_{\mathrm{Pl}}}{8}, \qquad \boxed{\Delta_t^2(q) \simeq 128\frac{P_{\mathrm{scr},*}}{P_{\mathrm{Pl}}} = \frac{16}{N_{S,*}} = \frac{16\ell_{\mathrm{P}}^2}{\pi L_{\kappa,*}^2}.} \tag{210}$$

The first equality is an exact identity of the stated background assignments; the spectrum retains the leading slow-roll accuracy of Eq. (208). Larger screen pressure corresponds to a smaller inflationary horizon, lower horizon entropy, and greater tensor power. Fluctuation *amplitude*, rather than power, scales as $N_S^{-1/2}$. The entropy is that of the inflationary horizon at the relevant exit time, not today's cosmological horizon.

The inverse-entropy scaling is established in the literature, not a new claim of this construction. Brustein and Medved explicitly write $P_T(H) \sim 1/S_{\rm dS}$ in their Eq. (70) and compare with the standard inflationary result [70]. Their P_T denotes tensor *power*, and their entropy is dimensionless in their unit convention; that notation is unrelated to P_L or $P_{\rm scr}$. This citation records prior recognition of the scaling, without adopting their microscopic model or its numerical normalization. Entropy-based inflationary analyses also predate it [71]. The present form relates the standard spectrum to the mechanically defined force-per-area response rather than introducing a new tensor spectrum.

b. Tilt as pressure evolution. Let $\mathcal{N} = \ln(a/a_{\rm ref})$ be increasing elapsed e-folds and $\epsilon_H = -\dot{H}/H^2$. The background definitions give exactly

$$\frac{d\ln P_{\rm scr}}{d\mathcal{N}} = -2\epsilon_H, \qquad \frac{d\ln N_S}{d\mathcal{N}} = 2\epsilon_H. \tag{211}$$

At first order in slow roll, the tensor tilt becomes

$$n_t \equiv \frac{d\ln \Delta_t^2}{d\ln q} \simeq -2\epsilon_{H,*} = \left.\frac{d\ln P_{\rm scr}}{d\mathcal{N}}\right|_* = -\left.\frac{d\ln N_S}{d\mathcal{N}}\right|_*. \tag{212}$$

The standard $n_t \simeq -2\epsilon_H$ is used here [69]. Along exit times, $d\ln q = (1-\epsilon_H)d\mathcal{N}$; identifying these logarithmic increments is a first-order slow-roll step, not an exact all-orders statement about the spectrum. Tensor amplitude probes the pressure scale, while tilt probes its fractional evolution. A slowly falling screen pressure produces a red tensor tilt, or equivalently a growing horizon entropy.

c. Scalar normalization and inference. At a reference wave number $q_\star$, define $A_s = \Delta^2_{\mathcal{R}}(q_\star)$ and $r = \Delta_t^2(q_\star)/A_s$. Then

$$P_{{\rm scr},*} \simeq \frac{rA_s}{128} P_{\rm Pl}, \qquad N_{S,*} \simeq \frac{16}{rA_s}. \tag{213}$$

These are alternative parameterizations of the same inferred inflationary scale; they are not independent measurements of a local pressure. Under the additional assumptions of canonical single-field slow-roll inflation with the standard scalar vacuum spectrum, $r \simeq 16\epsilon_{H,*}$ [69], and therefore

$$r \simeq -8 \left.\frac{d\ln P_{\rm scr}}{d\mathcal{N}}\right|_* = 8 \left.\frac{d\ln N_S}{d\mathcal{N}}\right|_*, \qquad r \simeq -8n_t. \tag{214}$$

The scalar assumption is needed for this last relation, not for the pressure substitution in the tensor amplitude.

The pressure in these expressions is the homogeneous horizon response, not a proposed tensor-noise source. No molecular interpretation of N_S is required. A boundary derivation of the fluctuations would still have to reproduce the tensor two-point function, normalization, and state dependence from specified dynamical and quantum inputs. Changing the tensor action, initial state, entropy law, or horizon-rate prescription can alter the correspondence. Equations (210)–(214) are consequently a consistency target and a mechanical interpretation of established inflationary predictions, not a derivation of those predictions from the classical boundary potential alone.

VII. INTEGRABILITY AND FINITE DYNAMICAL BALANCE

A. When the scale leg is and is not a state energy

On the extended boundary source space, define the scale one-form

$$\alpha_L \equiv -F_L\, \boldsymbol{\delta} L_{\rm s} = \frac{Ac^2}{8\pi G}\, \boldsymbol{\delta}\kappa. \tag{215}$$

Its phase-space exterior derivative is

$$\boxed{\mathbf{d}_\Gamma \alpha_L = \frac{c^2}{8\pi G} \boldsymbol{\delta} A \wedge \boldsymbol{\delta}\kappa = -P_L \boldsymbol{\delta} A \wedge \boldsymbol{\delta} L_{\rm s}.} \tag{216}$$

Therefore there is no universal state function $H_L(A, L_{\rm s})$ satisfying $\boldsymbol{\delta} H_L = \alpha_L$ when area and scale vary independently. Here "independently" means virtual source variations in the extended boundary ensemble, not a claim that every physical history explores an arbitrary two-dimensional path. The principal Schwarzschild and Cai–Kim families are one-dimensional and link A to L_κ; Rindler also supplies an operational setting in which patch area and observer acceleration can be controlled separately. The nonzero curl does not disqualify the term as work–ordinary $P\,dV$ is generally nonexact–and does not qualify the pressure law on a physical one-parameter family. It rules out only a universally separate stored scale energy on the unrestricted source space.

The complete parent variation remains exact because

$$\mathbf{d}_\Gamma(\Pi_\partial \boldsymbol{\delta} A) = -\mathbf{d}_\Gamma \alpha_L, \qquad \boldsymbol{\delta} U_\partial = \Pi_\partial \boldsymbol{\delta} A + \alpha_L. \tag{217}$$

On a one-dimensional physical family, or any leaf $L_\mathrm{s} = L_\mathrm{s}(A)$, the pullback of the two-form vanishes and the scale leg is integrable along that leaf. Ashtekar, Fairhurst, and Krishnan obtained the equivalent condition $\boldsymbol{\delta}\kappa \wedge \boldsymbol{\delta} A = 0$ for isolated-horizon Hamiltonian evolution [42].

For spherical Hayward horizons, using

$$\kappa_H^\mathrm{s} = \frac{c^2}{2R} - \frac{4\pi G R}{c^2} W \tag{218}$$

in Eq. (216) gives the exact reduced obstruction

$$\boxed{\mathbf{d}_\Gamma \alpha_L = -\boldsymbol{\delta} V \wedge \boldsymbol{\delta} W.} \tag{219}$$

The isolated scale leg is therefore integrable on restricted leaves such as fixed W, $W = W(V)$, or a single history, but not for generic independent variations of screen size and equation of state.

B. Finite constant-force law for dynamical black holes

Ashtekar, Paraizo, and Shu proved an exact nonperturbative charge–flux law for quasi-local dynamical black-hole horizons that can be arbitrarily far from equilibrium [43]. Their preferred equilibrium projection assigns each marginal cut the Kerr surface gravity $\bar{\kappa}(A, J)$. In their geometric units the phase-space charge is $Q_\mathrm{PS} = \bar{\kappa} A/(8\pi G)$ and obeys

$$\Delta Q_\mathrm{PS} = \mathfrak{F}^{(\xi)}_\mathrm{gws} + \mathfrak{F}^{(\xi)}_\mathrm{matt}. \tag{220}$$

Restoring acceleration units and defining

$$L_\mathrm{APS} = \frac{c^2}{\bar{\kappa}}, \qquad U_\mathrm{APS} = \frac{A c^2 \bar{\kappa}}{8\pi G}, \qquad F_L^\mathrm{APS} = \frac{A \bar{\kappa}^2}{8\pi G}, \tag{221}$$

gives the exact acceleration-length representation

$$\boxed{\Delta U_\mathrm{APS} = \int_{S_1}^{S_2} \left(\Pi_\partial\, dA - F_L^\mathrm{APS} dL_\mathrm{APS}\right) = E^{(\xi)}_\mathrm{gws} + E^{(\xi)}_\mathrm{matt}.} \tag{222}$$

APS supply the second equality; the first is the present source representation. Matter and gravitational fluxes do not separately map onto the area and scale legs.

In the nonrotating sector the equilibrium projection is Schwarzschild:

$$\bar{\kappa} = \frac{c^2}{2R}, \qquad L_\mathrm{APS} = 2R, \qquad U_\mathrm{APS} = \frac{c^4 R}{4G}, \qquad F_L^\mathrm{APS} = \frac{c^4}{8G}. \tag{223}$$

The force is exactly constant. Consequently the finite physical-process law becomes

$$\boxed{E^{(\xi)}_\mathrm{gws} + E^{(\xi)}_\mathrm{matt} = \Delta U_\mathrm{APS} = F_L^\mathrm{APS}\, \Delta L_\mathrm{APS}.} \tag{224}$$

This is a finite charge–flux law expressed in the projected acceleration length. It does not identify that length with a proper material displacement during the dynamical process.

APS normalize the nonrotating horizon mass as $E_\mathrm{DH} = 2U_\mathrm{APS}$. Since $U_\mathrm{APS} = T_\mathrm{proj} S = F_L^\mathrm{APS} L_\mathrm{APS}$ on every projected cut, with $T_\mathrm{proj} = \hbar\bar{\kappa}/(2\pi k_B c)$,

$$\boxed{\Delta(T_\mathrm{proj} S) = \Delta(F_L L) = \Delta U_\mathrm{APS} = \frac{1}{2}\Delta E_\mathrm{DH}.} \tag{225}$$

The two projected Euler products have the same finite change because both equal $U_{\rm APS}$. The nontrivial additional result is that APS relate this charge change to finite matter and gravitational fluxes. This does not establish independent energy reservoirs, assign either product to one flux carrier, or imply a KMS state during arbitrary evolution. $T_{\rm proj}$ is the temperature parameter of the preferred equilibrium projection.

For rotating cuts, A, J, $\bar{\kappa}$, and $\bar{\Omega}$ are linked by the Kerr projection and the rotational channel must be retained. The constant-force simplification is specific to the nonrotating four-dimensional sector.

VIII. STATIONARY WALD-ENTROPY AND HIGHER-CURVATURE EXTENSION

The acceleration-length idea has a clean stationary extension, but two notions must be separated: the universal $S_W\,\delta T$ channel and the specific membrane pressure of a higher-curvature theory.

Chakraborty and Padmanabhan showed that the null-surface variation of the Lanczos–Lovelock surface term can again be organized as $S_W\delta T$ and $T\delta S_W$, with S_W the appropriate Wald entropy [44]. Holding Wald entropy fixed and using $T=\hbar c/(2\pi k_B L_\kappa)$ gives

$$S_W\,dT=-F_L^{(W)}dL_\kappa,\qquad \boxed{F_L^{(W)}=\frac{TS_W}{L_\kappa},\qquad P_{L,SdT}^{(W)}=\frac{F_L^{(W)}}{A}=\frac{\hbar c}{2\pi k_B L_\kappa^2}s_W,} \tag{226}$$

where $s_W=S_W/A$. For Einstein gravity, $s_W=k_B/(4\ell_{\rm P}^2)$ and Eq. (226) reduces exactly to $\kappa^2/(8\pi G)$.

The membrane pressure in a pure mth-order Lanczos–Lovelock theory has an additional theory- and dimension-dependent factor. Kolekar and Kothawala found, for the isotropic horizon sector with maximally symmetric spatial cuts used in their derivation,

$$\frac{\Pi_\partial^{(m)}A}{T}=\frac{D-2m}{D-2}S_W^{(m)}. \tag{227}$$

At fixed Wald entropy density and couplings, its acceleration-length susceptibility is therefore

$$\boxed{P_{L,\rm mem}^{(m)}=-\left(\frac{\partial\Pi_\partial^{(m)}}{\partial L_\kappa}\right)_{s_W}=\frac{D-2m}{D-2}\frac{\hbar c}{2\pi k_B L_\kappa^2}s_W^{(m)}.} \tag{228}$$

For a sum of Lovelock densities, the corresponding stationary membrane response is the sum of the weighted entropy densities. Einstein gravity is the $m=1$ case and the factor equals one. In the critical dimension $D=2m$, the pure Lovelock term is topological: it can contribute a global Wald entropy while its local membrane pressure factor vanishes. This is a useful check rather than a paradox [45].

The distinction between Eqs. (226) and (228) is important. The first is the response of the thermodynamic $S_W\delta T$ channel. The second is the derivative of the actual Lanczos–Lovelock membrane pressure. They coincide in Einstein gravity but not universally in higher-curvature theories. Moreover, fixed area need not mean fixed Wald entropy. Along a physical family for which S_W changes with L_κ,

$$-\frac{d(TS_W)}{dL_\kappa}=\frac{TS_W}{L_\kappa}-T\frac{dS_W}{dL_\kappa}, \tag{229}$$

so the second term belongs to the entropy/area sector rather than the fixed-entropy scale response.

For stationary $f(R)$ gravity with f_R effectively constant over the source variation, $s_W=f_Rk_B/(4\ell_{\rm P}^2)$ and the Wald-channel result becomes

$$P_{L,SdT}^{f(R)}=f_R\frac{\kappa^2}{8\pi G}. \tag{230}$$

Generic variations of f_R and generic dynamical higher-curvature horizons add derivative and non-Newtonian membrane terms. A complete covariant null-source proof in those theories remains separate work. The result established here is the stationary Wald-channel lift and its precise relation to the known Lovelock membrane equation of state.

IX. GENERALITY, ANTECEDENTS, AND INTRINSIC SIGNIFICANCE

A. Common potential, canonical scope, and physical realizations

Horizon boundary mechanics names the mechanics of the specified Einstein horizon boundary potential. It does not assert a complete theory of every boundary response. The same potential and normal-rate prescription give the same mechanical normal form; canonical support, physical displacement, and geometric work retain their boundary conditions.

The covariant realization has three levels:

1. The generic covariant Einstein null scalar source is $\mu = \widehat{\kappa} + \theta/2$.
2. On a clock-fixed non-expanding null sector, $\mu = \widehat{\kappa}$ and the branchwise coordinate change to $L_{\mathrm{s}} = 1/\widehat{\kappa}$ gives the exact Hamilton–Jacobi response $P_L = \kappa^2/(8\pi G)$.
3. In other sectors, an independently established normal-rate prescription and parent $A\kappa$ potential can support the same algebra, but not automatically the same reduced canonical theorem.

Squaring removes the orientation sign, not the clock dependence. Extremal or trace-boundary points with $\kappa = 0$ must be described in the regular parent rate rather than the reciprocal chart.

The pullback and normal-balance results sharpen the physical hierarchy. The generic source μ is the broadest null variable. The linear surface pressure/tension Π_∂ appears directly in tangential and normal screen equations. The quadratic P_L is the normal susceptibility of that stress with respect to the mechanically privileged acceleration length. It is not more general than μ, but it is more directly analogous to ordinary three-dimensional pressure because it is force per area conjugate to the selected normal length. Its conversion to geometric volume work requires the corresponding displacement dictionary.

In D spacetime dimensions, if a spherical sector has $A \propto R^{D-2}$ and $L_\kappa \propto R$, then

$$F_L = AP_L \propto R^{D-4}. \tag{231}$$

The scale-independent force is therefore special to four spacetime dimensions. This dimensional fact is independent of the numerical normalization selected in a particular sector.

B. Compact theorem and canonical certification

The compact acceleration-length derivation and the full covariant analysis carry different kinds of information. Within the inherited standard horizon-mechanics sector, the short chain

$$U_\partial(A, L_{\mathrm{s}}) = \frac{Ac^4}{8\pi G L_{\mathrm{s}}}, \qquad dU_\partial = \frac{U_\partial}{A}\, dA - \frac{U_\partial}{L_{\mathrm{s}}}\, dL_{\mathrm{s}} \tag{232}$$

is already a self-contained conditional theorem. Its derivatives give Π_∂, $F_L = (U_\partial/c^2)\kappa$, $P_L = F_L/A$, and the pressure-work form; Rindler, Schwarzschild, and Cai–Kim FRW then follow after their standard sector dictionaries are supplied. The clock, generator normalization, relational cut, and non-expanding or equilibrium assumptions are inherited from the same horizon mechanics that defines the parent potential, membrane stress, and Hawking temperature.

The role of the full construction is canonical grounding rather than algebraic completion. In the stated null-boundary sector it acts as a proof certificate for the compact result: it derives the admissible source variation from the mixed null symplectic potential, reproduces the same force from the independent area–boost corner action, and records the clock, gauge, sign, zero-surface-gravity, integrability, prescription, and dynamical limits. Some extensions are strongly generated by the compact normal form, while others–notably the off-family integrability obstruction, finite APS support, and higher-curvature distinctions–supply additional information rather than merely expanding the chain rule.

This separation is useful conceptually. The structural insight is the identification of acceleration length as the normal mechanical coordinate of the $A\kappa$ boundary potential. The formal analysis shows exactly how that insight is realized by general relativity, where it is inherited from standard horizon mechanics, and where it cannot be extended without further input.

C. Relation to prior work

The ingredients on which the construction rests are established: area–boost conjugacy; local near-horizon A/ℓ energy; the $A\kappa$ Kodama–Noether charge; the $T\delta S + S\delta T$ surface-term split; the null scalar pair and mixed boundary conditions; the Damour–Navier–Stokes membrane pressure; Freidel and Yokokura's Young–Laplace screen equation; Hayward's unified first law; the Cai–Kim apparent-horizon construction; and the APS finite dynamical charge–flux theorem [11–13, 16–20, 24, 25, 28, 37, 38, 43].

The novelty is not the existence of the $A\kappa$ potential, the $S\,dT$ channel, the linear membrane pressure, the Bekenstein–Hawking factor four, the Young–Laplace equation, or the APS charge. The added construction is the acceleration-length source representation and the structure it exposes:

1. the action and corner derivations of $P_L = \kappa^2/(8\pi G)$ and $F_L = AP_L$;
2. the direct pressure-work form $-P_L\vartheta_V^{(\delta)}$;
3. the signed pullback relation $D_A\Pi_\partial = -P_L D_A L_{\rm s}$ (equivalently $D_A|\Pi_\partial| = -P_L D_A L_\kappa$) and the normal balance $\Delta p - P_L L_{\rm s} H = 0$;
4. the mechanical–thermal bridge laws $P_L \propto L_\kappa^{-2}$ and $T \propto L_\kappa^{-1}$, together with the boundary phase/Compton relations $k_BT/F_L = \bar{\lambda}_\partial/(2\pi) = L_\kappa/N_S$, $\beta U_\partial = N_S$, and $\omega_\partial/\omega_1^{\rm E} = N_S/(2\pi)$;
5. the Schwarzschild Euler–Smarr and formal free-energy identities;
6. the positive Cai–Kim FRW screen pressure and compact first law;
7. the response functions and integrability criterion; and
8. the nonrotating APS finite constant-force flux law and the stationary Wald-entropy lift.

The contribution belongs to the same scientific archetype as the membrane reformulation of horizon dynamics, Iyer–Wald's Noether-charge organization, Jacobson's equation-of-state derivation, Hayward's unified first law, Padmanabhan's horizon thermodynamic identity, Cai–Kim's FRW construction, and the Frodden–Ghosh–Perez local first law: established gravity is reorganized so that a previously obscured thermodynamic or mechanical structure becomes explicit [10, 18, 21, 37–39]. This is a comparison of contribution type, not a claim that the present work has already attained the historical reach of those papers.

To our knowledge, prior work has not combined the mixed null rate-source polarization, the branchwise coordinate $L_\kappa = c^2/|\kappa|$, the quadratic response $\kappa^2/(8\pi G)$, its force–length work, and the Rindler–Schwarzschild–FRW–APS dictionary developed here. Equivalent variables can be hidden by conventions, so the priority claim remains appropriately qualified.

The positive FRW screen-work result was publicly disclosed by the author on May 10, 2025: `https://x.com/JTT329/status/1921341959009902709`. Kong's distinct effective-pressure preprint was first submitted on July 23, 2025 [41]; its negative volume-slope convention is compared with the positive screen response in Sec. VI. The inflationary relations in Sec. VI H are separate consequences obtained by importing the established tensor spectrum, whose inverse-entropy scaling is not claimed as new.

X. CONCLUSION

The framework developed here is horizon boundary mechanics: the area and acceleration-length response of a specified Einstein boundary potential. Its primary canonical foundation is the null scalar source $\mu = \widehat{\kappa} + \theta/2$. On a clock-anchored non-expanding sector it reduces to surface gravity, the regular geometric rate of the selected normal flow; reciprocal acceleration length is its mechanical scale and an admissible branchwise source coordinate. The direct symplectic-potential and independent area–boost results are

$$\Theta_L^{(0)} = -\int d\tau \int_S \epsilon_S\, \frac{\kappa^2}{8\pi G}\delta L_{\rm s}, \qquad P_L = \frac{\kappa^2}{8\pi G}, \qquad F_L = AP_L. \tag{233}$$

For a uniform cut the term is $-F_L\delta L_{\rm s} = -P_L A\delta L_{\rm s}$. The force-per-area interpretation is therefore tied to the selected scale variation. The volume-valued one-form $\vartheta_V^{(\delta)}$ is exact at fixed area, but a geometric volume interpretation requires the sector-specific dictionary. These statements retain the specified rate-source polarization and boundary conventions.

Radius and acceleration length are complementary coordinates for this structure. In the spherical scalar sector the exact work transformation is $F_L dL_s = M_R\kappa\, dR + U_\partial d\alpha_R/\alpha_R$. The normal-length description is more general than the radius work with its ratio term omitted; it is equivalent to the complete radius representation. Schwarzschild and Cai–Kim FRW fix the ratio to two and one. Spherically completing Rindler boundary data with $R = L_\kappa$ reproduces the latter energy and pressure relations exactly. Its geometrical content is a transverse-area choice, rather than an observer-induced bulk mass. A spherical horizon's mass–radius constraint and its normal metric gradient explain separately why its radius and its acceleration length take their respective values.

The linear membrane pressure and the normal pressure are different but linked. In signed source variables,

$$\Pi_\partial = P_L L_{\mathrm{s}}, \qquad D_A \Pi_\partial = -P_L D_A L_{\mathrm{s}}; \tag{234}$$

while in positive magnitudes $|\Pi_\partial| = P_L L_\kappa$ and $D_A|\Pi_\partial| = -P_L D_A L_\kappa$. The differential relation is the precise pullback of the source-space response to a nonuniform horizon cut. The same identity rewrites the Young–Laplace screen constraint as $\Delta p - P_L L_{\mathrm{s}} H = 0$, or $\Delta p - P_L L_\kappa H = 0$ in the positive-κ orientation, supplying a normal force balance under the matched screen-rate conventions.

Where a Hawking–Unruh/KMS state exists, acceleration length is also the reduced Euclidean thermal length:

$$\boxed{c\Delta\tau_E = 2\pi L_\kappa, \qquad L_\kappa = \frac{c^2}{|\kappa|} = \frac{c}{\omega_1^{\mathrm{E}}} = \frac{\hbar c}{2\pi k_B T}, \qquad P_L(L_\kappa) = \frac{c^4}{8\pi G L_\kappa^2}.} \tag{235}$$

The Euclidean/KMS relation is established; the added synthesis is that the same scale is the mechanical source coordinate of the classical Einstein pressure. Pressure supplies the boundary energy density $P_L L_\kappa$, temperature supplies its thermal conversion, and their ratio reconstructs one entropy nat–one unit of S/k_B–per $4\ell_{\mathrm{P}}^2$. An informational reading is possible, but the formulas do not specify a microscopic state count or an exterior observable algebra. The entropy identities use the standard Einstein normalization. Defining the boundary Compton-equivalent reduced length $\bar{\lambda}_\partial = \hbar c/U_\partial$ gives the further exact relations $k_B T/F_L = \bar{\lambda}_\partial/(2\pi) = L_\kappa/N_S$ and $2\pi L_\kappa = N_S \bar{\lambda}_\partial$. If U_∂ is represented by a quantum Hamiltonian generator, one may additionally write $\bar{\lambda}_\partial = c/\omega_\partial$ and $\omega_\partial/\omega_1^{\mathrm{E}} = N_S/(2\pi)$. In all cases $S/k_B = \beta U_\partial$ is the dimensionless Euclidean thermal exponent of the boundary potential over one period, or equivalently the number of reduced inverse-energy intervals around the thermal circumference. This inverse-energy/thermal-scale interpretation puts the Euler relation in Bekenstein form and shows that the clock-invariant ratio of the quantum inverse-energy scale to the horizon scale is $8\pi\ell_{\mathrm{P}}^2/A$.

Schwarzschild realizes the exact Euler–Smarr identities

$$Mc^2 = TS + F_L L_\kappa, \qquad TS = F_L L_\kappa = \mathcal{F}_{\mathrm{H}}^{\mathrm{on}} = \frac{Mc^2}{2}. \tag{236}$$

The two products are the same function of the extended sources, while the area and scale derivatives give distinct responses. Their exact gravitational scaling balance motivates a stationary virial test; a literal kinetic/potential decomposition remains to be derived. With $L_\kappa = 2R_H$, the scale work is $F_L dL_\kappa = (M\kappa_\infty) dR_H = 2P_L dV_H$. Thus the radius force is $M\kappa_\infty = 2F_L$ and the radius-work pressure is $2P_L$, with the full area leg retained. More generally, spherical cuts give $U_\partial = F_R R = 3P_R V$ with $F_R = M_R\kappa$, $M_R = c^2 R/(2G)$, and $P_R = \alpha_R P_L$. The pure scale-work form $F_R\, dR$ requires fixed $\alpha_R = L_{\mathrm{s}}/R$; otherwise it acquires $U_\partial\, d\alpha_R/\alpha_R$. This makes the Smarr homogeneity explicit without adding an independent mass constraint. The same scalar symplectic structure gives $\Omega^{(0)}/\Delta\tau = 2\delta R \wedge \delta F_R$. For Kerr, $U_\partial = M\kappa r_+$ is exact in the Boyer–Lindquist radius, and the pure $M\kappa\, dr_+$ scale work holds at fixed dimensionless spin; the rotational mass-work channel remains.

Cai–Kim FRW gives the direct areal-volume application, separately calibrated from the non-expanding null theorem: $P_L = P_{\mathrm{scr}} = \rho/3$ and $dE_A = T_{\mathrm{CK}} dS_A - P_{\mathrm{scr}} dV_A$. In pressure variables the Friedmann system reads $R_A^{-2} = \kappa_{\mathrm{E}} P_{\mathrm{scr}}$, $2\ddot{a}/(ac^2) = -\kappa_{\mathrm{E}}(P_{\mathrm{scr}} + p_{\mathrm{m}})$, and $\dot{P}_{\mathrm{scr}} + H(3P_{\mathrm{scr}} + p_{\mathrm{m}}) = 0$. The positive screen pressure fixes the transverse horizon scale; its sum with matter pressure fixes cosmic acceleration. Full Hayward surface gravity measures the complementary trace-sensitive contrast, $\kappa_H^{\mathrm{s}} = -(c^2 R_A/4)\kappa_{\mathrm{E}}(P_{\mathrm{scr}} - p_{\mathrm{m}})$, and admits the exact surface-density form $\kappa_H^{\mathrm{s}} = -2\pi G\Sigma_{\mathrm{tr}}$ with $\Sigma_{\mathrm{tr}} = (\rho - 3p_{\mathrm{m}})V_A/(c^2 A_A)$ before obeying the HMA redistribution law. Beyond equilibrium, the scale leg is generally nonintegrable under independent virtual source variations, although Schwarzschild and Cai–Kim physical families link area and scale and the complete parent charge remains exact. For nonrotating APS dynamical horizons, the same structure becomes the finite constant-force law

$$E_{\mathrm{gws}}^{(\xi)} + E_{\mathrm{matt}}^{(\xi)} = \frac{c^4}{8G}\, \Delta L_{\mathrm{APS}}. \tag{237}$$

A stationary higher-curvature extension replaces the Einstein entropy density by the appropriate Wald density, with a separate Lovelock-dependent membrane factor.

For spatially flat slow-roll inflation with the usual Einstein tensor vacuum, the standard tensor spectrum admits the representation $\Delta_t^2 \simeq 128P_{\rm scr}/P_{\rm Pl} = 16/N_S$ at mode exit. Its first-order tilt is the fractional pressure decrease per e-fold, $n_t \simeq d\ln P_{\rm scr}/d\mathcal{N} = -d\ln N_S/d\mathcal{N}$. Section VI H states the normalization, additional single-field scalar assumptions, and prior inverse-entropy literature. This connects boundary-mechanical variables to a standard quantum observable without deriving the tensor fluctuations from the pressure law or postulating a gas of horizon constituents.

The Planck-scale discussion is deliberately separated into benchmark levels. The clock-invariant equality $\bar{\lambda}_\partial = L_\kappa$ fixes the crossover area $8\pi\ell_{\rm P}^2$ without fixing the normal scale; a sector such as Cai–Kim or Schwarzschild then supplies its own $L_\kappa(R)$ relation. Adding $L_\kappa = \ell_{\rm P}$ gives the symmetric full-alignment benchmark with Planck boundary energy and force, whereas imposing $R = L_\kappa = \ell_{\rm P}$ under the unmodified Cai–Kim/Einstein normalization gives half-Planck boundary energy and force. These are transparent continuations of the present formulas, not claims that the classical or semiclassical coefficients survive unchanged in quantum gravity.

The common result is a mechanics of the Einstein boundary potential, with Rindler, stationary black-hole, and calibrated cosmological realizations. The canonical proof retains its boundary hypotheses; the shared normal form does not identify every evolving screen with that same boundary problem. Part I gives the generative normal form and its principal consequences; Part II provides the canonical proof certificate connecting that form to Einstein boundary theory and delimiting its extensions. Surface gravity is the regular geometric rate; acceleration length is its reciprocal mechanical scale and, where KMS thermality applies, the Euclidean thermal period expressed as a reduced light-travel length. Classically it is the coordinate conjugate to Einstein boundary pressure; semiclassically it fixes Hawking–Unruh temperature. The representation exposes this one scale as local Rindler geometry, Schwarzschild thermal–mechanical duality, FRW screen work, universal entropy density, and finite dynamical force–displacement balance, while stating exactly where clock choice, quantum state, horizon dynamics, higher curvature, and integrability limit the claim.

Appendix A: Signed and positive acceleration length

The canonical derivation uses the signed source $L_{\rm s} = c^2/\kappa$. On a fixed-sign branch define

$$\sigma \equiv \mathrm{sgn}(\kappa), \qquad L_\kappa = \frac{c^2}{|\kappa|} = |L_{\rm s}|, \qquad L_{\rm s} = \sigma L_\kappa. \tag{A1}$$

The signed boundary potential and surface stress are then

$$U_\partial = \sigma\frac{Ac^4}{8\pi G L_\kappa}, \qquad \Pi_\partial = P_L L_{\rm s} = \sigma P_L L_\kappa, \qquad |\Pi_\partial| = P_L L_\kappa. \tag{A2}$$

Their spatial derivatives obey

$$D_A\Pi_\partial = -P_L D_A L_{\rm s} = -\sigma P_L D_A L_\kappa, \qquad D_A|\Pi_\partial| = -P_L D_A L_\kappa, \tag{A3}$$

and the source work form becomes

$$\Theta_L^{(0)} = -\sigma\int d\tau \int_S \epsilon_S\, P_L \delta L_\kappa. \tag{A4}$$

P_L and all positive magnitudes are orientation independent, while the signed stress and work one-form carry σ. This is why the surface stress changes sign while the pressure does not: orientation reversal changes both Π_∂ and $L_{\rm s}$, leaving the signed ratio $\Pi_\partial/L_{\rm s} = P_L$ invariant. With the positive length one instead writes $|\Pi_\partial|/L_\kappa = P_L$. Neither reciprocal coordinate covers $\kappa = 0$.

Appendix B: FRW rate identities

Einstein's equations in the conventions of Eq. (158) are

$$H^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2}\rho, \tag{B1}$$

$$\dot{H} - \frac{kc^2}{a^2} = -\frac{4\pi G}{c^2}(\rho + p_{\rm m}), \tag{B2}$$

$$\dot{\rho} + 3H(\rho + p_{\rm m}) = 0. \tag{B3}$$

Differentiating Eq. (159) gives

$$\dot{R}_A = \frac{4\pi G}{c^4} H R_A^3 (\rho + p_{\rm m}), \tag{B4}$$

and hence

$$\frac{\dot{R}_A}{2HR_A} = \frac{2\pi G R_A^2}{c^4}(\rho + p_{\rm m}) = \frac{3}{4}(1+w), \tag{B5}$$

proving Eq. (181) for arbitrary k where $H \neq 0$. At $H = 0$ the regular non-ratio expressions in the main text apply.

The spherical surface-gravity identity

$$\kappa_H^{\rm s} = \frac{GE_A}{c^2 R_A^2} - \frac{4\pi G R_A}{c^2} W \tag{B6}$$

immediately gives Eq. (194); differentiating it and using Eq. (196) gives Eq. (197).

Appendix C: Timelike spherical boundary comparison and unresolved horizon matching

This appendix preserves a separate boundary-action investigation; it is not part of the principal horizon-pressure claim. The calculation below establishes a local Hamilton–Jacobi identity for one specified spherical timelike mixed ensemble, with the stated matter, clock, and endpoint conditions. It does *not* establish that this ensemble's response is the full canonical response to the original Hayward or Cai–Kim acceleration length on an independently moving horizon. An explicit FRW comparison gives different rates. The general matching problem remains open. Throughout this appendix set $c = 1$.

1. Action, matter, clock, and endpoint conditions

Let $\mathcal{B}$ be a smooth timelike spherical worldtube with induced metric

$$\gamma_{ij} dy^i dy^j = -N_B^2(t) dt^2 + R_B^2(t) d\Omega^2, \qquad A_B = 4\pi R_B^2. \tag{C1}$$

Its embedding may vary. Take Einstein gravity with minimally coupled scalar matter, the unreferenced Gibbons–Hawking–York term on $\mathcal{B}$, the corresponding cap terms, and the standard oriented joint terms:

$$I_D = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathcal{R}\epsilon + I_\phi + \frac{1}{8\pi G}\int_{\mathcal{B}} K\epsilon_{\mathcal{B}} + I_{\rm caps} + I_{\rm joints}. \tag{C2}$$

For example, $I_\phi = -\int[(\nabla\phi)^2/2 + V(\phi)]\epsilon$. There is no reference subtraction or additional source-dependent counterterm. Fix the pulled-back scalar boundary values and take local variations compactly supported away from the caps and their joints. The joint terms are included; their endpoint variations then vanish. If endpoints are varied, their area–boost contributions must be restored. No assertion of vanishing matter energy flux is made.

For embedding displacement χ^a, write δ_X for variation of the pullback to a fixed reference boundary. Thus $\delta_X \phi = X^*(\delta\phi + \mathcal{L}_\chi \phi) = 0$ is the matter condition, not $\delta\phi = 0$ at a fixed spacetime point. Normal motion contributes $2\chi_\perp K_{ij}$ to $\delta_X \gamma_{ij}$ in addition to the metric and tangential-reparameterization terms. It is included in the metric variation, not discarded by identifying the boundaries.

Use outward normal n^a, $K_{ij} = \gamma_i{}^a \gamma_j{}^b \nabla_a n_b$, and the unreferenced boundary stress

$$\tau^{ij} = \frac{K\gamma^{ij} - K^{ij}}{8\pi G}. \tag{C3}$$

The established on-shell action variation, including nonorthogonal boundary geometry, gives [58, 59]

$$\delta I_D\big|_{\mathcal{B}} = \frac{1}{2}\int_{\mathcal{B}} \epsilon_{\mathcal{B}} \tau^{ij} \delta_X \gamma_{ij} = \int dt\,[-E_B \delta_X N_B + N_B p_B \delta_X A_B]\,. \tag{C4}$$

Here $E_B = A_B \tau_{ij} u^i u^j$ and p_B is the isotropic surface stress, with force-per-length units after restoring c. Matter source variations vanish by the stated Dirichlet condition. Spherical symmetry removes tangential momentum and shear responses. The moving region may equivalently be pulled back to a fixed reference region; diffeomorphism covariance then includes the embedding variation in the pulled-back fields. The bulk field equations and constraints are used, so there is no additional independent on-shell shape term omitted from Eq. (C4) under these conditions.

2. The area-mixed action and its exact source response

Choose the boundary Legendre transform and define a rescaled surface-stress source

$$I_M = I_D - \int dt\, N_B A_B p_B, \qquad \zeta_B = -8\pi G N_B p_B. \tag{C5}$$

This is a spherical area-mixed polarization with a separate clock source; it is not an identification with the full three-dimensional York conformal ensemble. Before introducing ζ_B, the variation is

$$\delta I_M\big|_{\mathcal{B}} = \int dt[-E_B \delta_X N_B - A_B \delta_X (N_B p_B)]. \tag{C6}$$

The first term comes from varying the boundary lapse, whose conjugate is the Brown–York energy in this source convention. The second is the surface-stress source obtained by exchanging area for its conjugate. Thus the displayed coefficient $A_B/(8\pi G)$ in the next equation partly reflects the *definition* $\zeta_B = -8\pi G N_B p_B$. It is not, by itself, evidence that ζ_B is a horizon surface gravity. Brown–York boundary variation is established; this particular spherical Legendre transform and source normalization are choices made here for the comparison. Direct substitution gives

$$\boxed{\delta I_M\big|_{\mathcal{B}} = \int dt \left[-E_B \delta_X N_B + \frac{A_B}{8\pi G}\delta_X \zeta_B\right].} \tag{C7}$$

On a branch $\zeta_B \neq 0$, let $L_B = 1/\zeta_B$. Then

$$\boxed{\delta I_M\big|_{\mathcal{B}} = \int dt \left[-E_B \delta_X N_B - F_B \delta_X L_B\right], \quad F_B = \frac{A_B \zeta_B^2}{8\pi G}.} \tag{C8}$$

With the boundary clock N_B fixed, this is the complete local Hamilton–Jacobi boundary variation for the specified spherical ensemble and matter/endpoint conditions, wherever the on-shell action is defined. The response is $-(\delta I_{M,\mathrm{os}}/\delta L_B)_{N_B,\phi} = F_B$ and its force per area is $\mathcal{P}_B = \zeta_B^2/(8\pi G)$. As in the null theorem, source variations compare nearby boundary-value problems; conservative variations within one problem fix the selected sources. This variational identity does not assert existence or uniqueness of every initial-boundary value problem.

The surface stress is determined geometrically. Put

$$a_n = n_a u^b \nabla_b u^a, \qquad b_B = \frac{n^a \nabla_a R_B}{R_B}. \tag{C9}$$

The spherical extrinsic-curvature eigenvalues give

$$E_B = -\frac{A_B b_B}{4\pi G}, \qquad p_B = \frac{a_n + b_B}{8\pi G}, \qquad \boxed{\zeta_B = -N_B(a_n + b_B).} \tag{C10}$$

The factor $8\pi G$ in ζ_B removes the Einstein normalization, N_B converts the local stress contribution to the selected time normalization, and the minus sign records the chosen mixed-source orientation. This normalization makes the comparison transparent but does not by itself identify a horizon rate. The geometric eigenvalues provide the independent content: a_n is the normal acceleration of the timelike observers, and b_B measures normal transverse-size change per radius. The broader source is therefore a clock-weighted extrinsic boundary-stress rate, not surface gravity by definition.

The rescaled source therefore includes both normal acceleration and cut extrinsic curvature. It is not a priori a horizon surface gravity. Writing $\kappa_B = c^2\zeta_B$ would restore acceleration units and give $\mathcal{P}_B = \kappa_B^2/(8\pi G)$ in the corresponding clock convention, but would not supply the missing geometric identification.

In the language of the main paper, one can also form the auxiliary state function $U_B = A_B\zeta_B/(8\pi G)$ and $\Pi_B = \zeta_B/(8\pi G) = -N_B p_B$. Its differential is

$$dU_B = \Pi_B\, dA_B - F_B\, dL_B. \tag{C11}$$

This exact algebraic identity is distinct from the action variation: U_B is not generally the Brown–York energy E_B, the Misner–Sharp energy, or the horizon potential U_∂ for a prescribed κ_H^{s} or κ_{CK}. Likewise, $L_B = 1/\zeta_B$ is a reciprocal boundary-stress coordinate, not an independently measured proper displacement. It agrees with the original c^2/κ only where the chosen geometry, orientation, and clock establish that $\zeta_B = \kappa/c^2$. No replacement of the main paper's definition of acceleration length is proposed.

3. Explicit flat-FRW moving-screen check

In cosmic-time/areal-radius coordinates, use

$$ds^2 = -(1 - H^2R^2)dt^2 - 2HR\,dt\,dR + dR^2 + R^2 d\Omega^2. \tag{C12}$$

For an arbitrary timelike embedding $R = R_B(t)$, define its velocity relative to the comoving flow and its lapse by

$$v = \dot{R}_B - HR_B, \qquad N_B = \sqrt{1 - v^2}, \qquad \gamma_v = (1 - v^2)^{-1/2}. \tag{C13}$$

The comoving orthonormal frame gives

$$a_n = \gamma_v^3 \dot{v} + \gamma_v H v, \qquad b_B = \frac{\gamma_v(1 + HR_B v)}{R_B}. \tag{C14}$$

Substitution in Eq. (C10) yields the exact source

$$\zeta_B = -\left[\frac{\dot{v}}{1 - v^2} + 2Hv + \frac{1}{R_B}\right]. \tag{C15}$$

On the timelike part of the actual apparent horizon, $R_A = 1/H$, put $q = \dot{R}_A = -\dot{H}/H^2$. Then $v = q - 1$ and $0 < q < 2$, so

$$\boxed{\zeta_B = \frac{1 - 2q}{R_A} - \frac{\dot{q}}{q(2 - q)}, \quad \kappa_{\rm CK} = \frac{1}{R_A}, \quad \kappa_H^{\rm s} = \frac{q/2 - 1}{R_A}.} \tag{C16}$$

For an Einstein–FRW background $q = 3(1 + w)/2$. A homogeneous scalar can realize a background with constant $w = 0$; then

$$\zeta_B = -\frac{2}{R_A}, \qquad \kappa_{\rm CK} = \frac{1}{R_A}, \qquad \kappa_H^{\rm s} = -\frac{1}{4R_A}. \tag{C17}$$

An orientation reversal changes the sign of ζ_B, not its magnitude. This is an explicit demonstration that the natural rate of this specified timelike mixed action differs from both cosmological rates. It is not a disproof of the separate Cai–Kim screen identity or the HMA Noether potential, nor a no-go theorem for every possible boundary ensemble. The endpoints $q = 0, 2$ are null and require a limit or a null action; the timelike expression must not be used there without qualification.

The inclusion of normal motion can be checked directly at the action level. For a homogeneous minimally coupled scalar on an Einstein–FRW solution, $p = -(2\dot{H} + 3H^2)/(8\pi G)$. Integrating the full on-shell Einstein-plus-scalar bulk action inside $R_B(t)$ and adding the timelike GHY term gives, up to the fixed cap and joint terms,

$$I_D = \frac{1}{G}\int dt \left\{\frac{R_B^3}{6}(\dot{H} + 3H^2) + \frac{R_B^2}{2}\left[\frac{\dot{v}}{1 - v^2} + 3Hv + \frac{2}{R_B}\right]\right\}. \tag{C18}$$

Vary $R_B(t)$ at fixed background and with compact time support; the pulled-back scalar values are fixed because the scalar is homogeneous. The resulting Euler–Lagrange shape variation agrees identically with Eq. (C4), including $\delta_X v = \delta\dot{R}_B - H\delta R_B$ and the induced lapse variation. This checks the embedding contribution using the complete bulk and boundary action, not only a change of thermodynamic coordinates.

For a stationary metric $ds^2 = -f(r)dt^2 + dr^2/f(r) + r^2 d\Omega^2$, the inner-boundary orientation gives $\zeta_B = f'/2 + f/r$. At a nonextremal horizon its limit is $f'(R_H)/2$, the Killing surface gravity in the selected time normalization. This establishes the stationary Schwarzschild limit in that orientation and clock normalization; it does not establish generic cosmological matching. For a de Sitter static patch, the outer-boundary orientation instead gives $\zeta_B = -f'/2 - f/r$, tending to $+1/R_H$ when $f = 1 - r^2/R_H^2$. This equals the positive Cai–Kim magnitude and the magnitude of the signed Hayward rate. The latter is $-1/R_H$ in the cosmological convention, so comparison of signed rates also requires matching orientation. These are limits of stationary stretched boundaries, not evaluations of Eq. (C16) at its null endpoint $q = 0$.

4. Relation to the FRW gravitational surface charges

A complementary calculation keeps the Einstein–Hilbert charge prescription instead of changing the boundary ensemble. In flat FRW, with $K = \partial_t$, fixed areal cut, and variations $\delta H(t)$, the Einstein potential and Noether charge give

$$\int_S Q_K = -\frac{R^3}{4G}(\dot{H} + 2H^2), \tag{C19}$$

$$\int_S \iota_K \theta_{\rm EH} = -\frac{R^3}{4G}(\delta\dot{H} + 8H\delta H), \tag{C20}$$

$$\int_S (\delta Q_K - \iota_K \theta_{\rm EH}) = \frac{HR^3}{G}\delta H = \delta E_{\rm MS}\big|_R. \tag{C21}$$

For $X = \partial_t + z(t)\partial_R$, with the displayed radial extension, the adjusted charge is

$$\int_S (\delta Q_X - Q_{\delta X} - \iota_X \theta_{\rm EH}) = \frac{R^2}{2G}(2HR - z)\delta H. \tag{C22}$$

Evaluating on $R_A = 1/H$, setting $z = \dot{R}_A$, and restricting to $\delta H = -\delta R_A/R_A^2$ gives $\kappa_H^{\rm s}\delta A/(8\pi G)$. These are restricted gravitational surface-charge identities, not replacements for the full timelike action variation above. They identify precisely where the Misner–Sharp energy and Hayward area coefficient enter.

Spatial curvature also matters for this charge prescription. With $b = k/a^2$, $D = 1 - bR^2 > 0$, $\dot{b} = -2Hb$, and $K = \sqrt{D}\,\partial_t$, the same fixed-cut gravitational calculation gives

$$\int_S (\delta Q_K - Q_{\delta K} - \iota_K \theta_{\rm EH}) = \delta\left[\frac{R^3(H^2+b)}{2G}\right]_R + \frac{R^5(b-\dot{H})}{8GD}\delta b. \tag{C23}$$

No cancellation of the extra term by matter or other boundary choices is claimed. The all-curvature HMA and constrained-screen identities in the main text are unaffected.

This result also clarifies the relation to the stretched-Carrollian potential of Freidel and Jai-akson, whose scalar sector retains clock, area, and stretching responses even in spherical symmetry [60]. Here the clock response is displayed in Eq. (C7); the timelike metric packages the stretching and motion into N_B, a_n, and b_B. Their more general source space has not been identified with a single Hayward-rate variation. Establishing that identification would require the corresponding action, rigging, and source restrictions explicitly.

The established result of this appendix is the displayed local variational identity for the chosen ensemble, together with the explicit action and surface-charge checks. The unresolved step is a canonical calculation for the intended physical horizon prescription whose complete response can be identified with the original $\kappa_H^{\rm s}$ or $\kappa_{\rm CK}$ while all other sources and embedding contributions are treated consistently. The unequal rates above prevent that identification for this ensemble on a generic FRW screen; they are not a no-go theorem for other ensembles. No new universal acceleration length or horizon-pressure law is claimed here.

---

[1] J. D. Bekenstein, Black holes and entropy, Phys. Rev. D **7**, 2333 (1973).
[2] J. D. Bekenstein, Universal upper bound on the entropy-to-energy ratio for bounded systems, Phys. Rev. D **23**, 287 (1981).
[3] E. P. Verlinde, On the origin of gravity and the laws of Newton, JHEP **04**, 029 (2011), arXiv:1001.0785.
[4] M. J. Lake and B. Carr, Does Compton/Schwarzschild duality in higher dimensions exclude TeV quantum gravity?, Int. J. Mod. Phys. D **27**, 1930001 (2018), arXiv:1808.08386, doi:10.1142/S0218271819300015.
[5] J. F. Donoghue, General relativity as an effective field theory: The leading quantum corrections, Phys. Rev. D **50**, 3874 (1994).
[6] S. W. Hawking, Particle creation by black holes, Commun. Math. Phys. **43**, 199 (1975).
[7] W. G. Unruh, Notes on black-hole evaporation, Phys. Rev. D **14**, 870 (1976).
[8] R. M. Wald, The entropy of black holes, Gen. Relativ. Gravit. **57**, 87 (2025).
[9] R. M. Wald, The thermodynamics of black holes, Living Rev. Relativ. **4**, 6 (2001), arXiv:gr-qc/9912119.
[10] T. Jacobson, Thermodynamics of spacetime: The Einstein equation of state, Phys. Rev. Lett. **75**, 1260 (1995), arXiv:gr-qc/9504004.
[11] F. Hopfmüller and L. Freidel, Gravity degrees of freedom on a null surface, Phys. Rev. D **95**, 104006 (2017), arXiv:1611.03096.
[12] F. Hopfmüller and L. Freidel, Null conservation laws for gravity, Phys. Rev. D **97**, 124029 (2018), arXiv:1802.06135.

[13] G. Odak, A. Rignon-Bret, and S. Speziale, General gravitational charges on null hypersurfaces, JHEP **12**, 038 (2023), arXiv:2309.03854.
[14] V. Chandrasekaran, E. E. Flanagan, and K. Prabhu, Symmetries and charges of general relativity at null boundaries, JHEP **11**, 125 (2018), arXiv:1807.11499.
[15] V. Chandrasekaran and E. E. Flanagan, Horizon phase spaces in general relativity, JHEP **07**, 017 (2024), arXiv:2309.03871.
[16] S. Carlip and C. Teitelboim, The off-shell black hole, Class. Quantum Grav. **12**, 1699 (1995), arXiv:gr-qc/9312002.
[17] E. Bianchi and W. Wieland, Horizon energy as the boost boundary term in general relativity and loop gravity, arXiv:1205.5325.
[18] E. Frodden, A. Ghosh, and A. Perez, Quasilocal first law for black hole thermodynamics, Phys. Rev. D **87**, 121503(R) (2013), arXiv:1110.4055.
[19] S. A. Hayward, S. Mukohyama, and M. C. Ashworth, Dynamic black-hole entropy, Phys. Lett. A **256**, 347 (1999), arXiv:gr-qc/9810006.
[20] K. Parattu, B. R. Majhi, and T. Padmanabhan, Structure of the gravitational action and its relation with horizon thermodynamics and emergent gravity paradigm, Phys. Rev. D **87**, 124011 (2013), arXiv:1303.1535.
[21] V. Iyer and R. M. Wald, Some properties of Noether charge and a proposal for dynamical black hole entropy, Phys. Rev. D **50**, 846 (1994), arXiv:gr-qc/9403028.
[22] R. M. Wald and A. Zoupas, A general definition of conserved quantities in general relativity and other theories of gravity, Phys. Rev. D **61**, 084027 (2000), arXiv:gr-qc/9911095.
[23] M. R. Visser and Z. Yan, Properties of dynamical black hole entropy, JHEP **10**, 029 (2024), arXiv:2403.07140.
[24] T. Damour, Surface effects in black-hole physics, in *Proceedings of the Second Marcel Grossmann Meeting on General Relativity*, edited by R. Ruffini (North-Holland, 1982), p. 587.
[25] R. H. Price and K. S. Thorne, Membrane viewpoint on black holes: Properties and evolution of the stretched horizon, Phys. Rev. D **33**, 915 (1986).
[26] K. S. Thorne, R. H. Price, and D. A. Macdonald, eds., *Black Holes: The Membrane Paradigm* (Yale University Press, 1986).
[27] S. Kolekar and T. Padmanabhan, Action principle for the fluid–gravity correspondence and emergent gravity, Phys. Rev. D **85**, 024004 (2012), arXiv:1109.5353.
[28] L. Freidel and Y. Yokokura, Non-equilibrium thermodynamics of gravitational screens, Class. Quantum Grav. **32**, 215002 (2015), arXiv:1405.4881.
[29] J. C. Maxwell, Attraction, *Encyclopaedia Britannica*, 9th ed., Vol. III (A. & C. Black, Edinburgh, 1875), pp. 63–65.
[30] J. L. Synge, Newtonian gravitational field theory, Nuovo Cimento B **8**, 373 (1972).
[31] R. Beig, G. W. Gibbons, and R. M. Schoen, Gravitating opposites attract, Class. Quantum Grav. **26**, 225013 (2009), arXiv:0907.1193.
[32] J. F. Cariñena, F. Falceto, and M. F. Rañada, A geometric approach to a generalized virial theorem, J. Phys. A **45**, 395210 (2012), arXiv:1209.4584.
[33] G. W. Gibbons and S. W. Hawking, Action integrals and partition functions in quantum gravity, Phys. Rev. D **15**, 2752 (1977).
[34] T. Matsubara, A new approach to quantum-statistical mechanics, Prog. Theor. Phys. **14**, 351 (1955).
[35] J. J. Bisognano and E. H. Wichmann, On the duality condition for a Hermitian scalar field, J. Math. Phys. **16**, 985 (1975); On the duality condition for quantum fields, J. Math. Phys. **17**, 303 (1976).
[36] V. E. Ambruş and M. N. Chernodub, Acceleration as a circular motion along an imaginary circle: Kubo–Martin–Schwinger condition for accelerating field theories in imaginary-time formalism, Phys. Lett. B **855**, 138757 (2024), arXiv:2308.03225.
[37] R.-G. Cai and S. P. Kim, First law of thermodynamics and Friedmann equations of FRW universe, JHEP **02**, 050 (2005), arXiv:hep-th/0501055.
[38] S. A. Hayward, Unified first law of black-hole dynamics and relativistic thermodynamics, Class. Quantum Grav. **15**, 3147 (1998), arXiv:gr-qc/9710089.
[39] T. Padmanabhan, Classical and quantum thermodynamics of horizons in spherically symmetric spacetimes, Class. Quantum Grav. **19**, 5387 (2002), arXiv:gr-qc/0204019.
[40] V. Faraoni, Cosmological apparent and trapping horizons, Phys. Rev. D **84**, 024003 (2011), arXiv:1106.4427.
[41] S.-B. Kong, Effective pressure of the FRW universe, Ann. Phys. **481**, 170163 (2025), doi:10.1016/j.aop.2025.170163.
[42] A. Ashtekar, S. Fairhurst, and B. Krishnan, Isolated horizons: Hamiltonian evolution and the first law, Phys. Rev. D **62**, 104025 (2000), arXiv:gr-qc/0005083.
[43] A. Ashtekar, D. E. Paraizo, and J. Shu, Thermodynamics of dynamical black holes beyond perturbation theory, Gen. Relativ. Gravit. **58**, 91 (2026), arXiv:2604.00170, doi:10.1007/s10714-026-03585-4. See also the companion letter, Thermodynamics of Black Holes, Far from Equilibrium, Phys. Rev. Lett. **136**, 251405 (2026), doi:10.1103/3c1r-v8f1, arXiv:2512.11659.
[44] S. Chakraborty and T. Padmanabhan, Geometrical variables with direct thermodynamic significance in Lanczos–Lovelock gravity, Phys. Rev. D **90**, 084021 (2014), arXiv:1408.4791.
[45] S. Kolekar and D. Kothawala, Membrane paradigm and horizon thermodynamics in Lanczos–Lovelock gravity, JHEP 02 (2012) 006, arXiv:1111.1242, doi:10.1007/JHEP02(2012)006.
[46] R. Dey, S. Liberati, and D. Pranzetti, The black hole quantum atmosphere, Phys. Lett. B **774**, 308 (2017), arXiv:1701.06161.
[47] R. Dey, S. Liberati, Z. Mirzaiyan, and D. Pranzetti, Black hole quantum atmosphere for freely falling observers, Phys. Lett. B **797**, 134828 (2019), arXiv:1906.02958.
[48] G. W. Gibbons and S. W. Hawking, Cosmological event horizons, thermodynamics, and particle creation, Phys. Rev. D **15**, 2738 (1977).
[49] S. Stotyn, K. Schleich, and D. M. Witt, Observer dependent horizon temperatures: A coordinate-free formulation of

Hawking radiation as tunneling, Class. Quantum Grav. **26**, 065010 (2009), arXiv:0809.5093.
[50] T. Wang, Cosmological horizons as new examples of membrane paradigm, Class. Quantum Grav. **32**, 195006 (2015), arXiv:1411.6445.
[51] M. Oltean, H. Bazrafshan Moghaddam, and R. J. Epp, Energy of cosmological spacetimes and perturbations: a quasilocal approach, Class. Quantum Grav. **38**, 085012 (2021), arXiv:2006.10068.
[52] P. Beltracchi, P. Gondolo, and E. Mottola, Surface stress tensor and junction conditions on a rotating null horizon, Phys. Rev. D **105**, 024001 (2022), arXiv:2103.05074.
[53] L. Freidel and P. Jai-akson, Carrollian hydrodynamics and symplectic structure on stretched horizons, JHEP **05**, 135 (2024), arXiv:2211.06415.
[54] Z.-W. Wang and S. L. Braunstein, Hawking radiation as a flux of Noether charge, Phys. Rev. D **114**, 044065 (2026), doi:10.1103/t75q-7bp9.
[55] A. Ashtekar, C. Beetle, and J. Lewandowski, Mechanics of rotating isolated horizons, Phys. Rev. D **64**, 044016 (2001), arXiv:gr-qc/0103026.
[56] L. Smarr, Mass formula for Kerr black holes, Phys. Rev. Lett. **30**, 71 (1973).
[57] J. M. Bardeen, B. Carter, and S. W. Hawking, The four laws of black hole mechanics, Commun. Math. Phys. **31**, 161 (1973).
[58] J. D. Brown, S. R. Lau, and J. W. York, Action and energy of the gravitational field, arXiv:gr-qc/0010024v3 (2002).
[59] G. Odak and S. Speziale, Brown–York charges with mixed boundary conditions, JHEP **11**, 224 (2021), arXiv:2109.02883.
[60] L. Freidel and P. Jai-akson, Geometry of Carrollian stretched horizons, arXiv:2406.06709v2 (2024), Sec. 5.1.
[61] C. A. R. Herdeiro, J. M. S. Oliveira, A. M. Pombo, and E. Radu, Virial identities in relativistic gravity: 1D effective actions and the role of boundary terms, Phys. Rev. D **104**, 104051 (2021), arXiv:2109.05027.
[62] C. W. Misner and D. H. Sharp, "Relativistic equations for adiabatic, spherically symmetric gravitational collapse," Phys. Rev. **136**, B571–B576 (1964).
[63] S. A. Hayward, "Gravitational energy in spherical symmetry," Phys. Rev. D **53**, 1938–1949 (1996), arXiv:gr-qc/9408002.
[64] V. Balasubramanian, B. Czech, B. D. Chowdhury, and J. de Boer, "The entropy of a hole in spacetime," JHEP **10**, 220 (2013), arXiv:1305.0856.
[65] H. Culetu, "On geodesics in spherical Rindler space," arXiv:2404.14621.
[66] D. Tong, *Statistical Physics*, Example Sheet 2 (February 2012), Problem 3, `https://www.davidtong.org/pdfs/teaching/statistical-physics/omg2.pdf`.
[67] D. Tong, *Statistical Physics*, Sec. 3.2, "Photons: Blackbody Radiation," University of Cambridge lecture notes, `https://www.davidtong.org/pdfs/teaching/statistical-physics/statphys3.pdf`.
[68] D. Tong, *Statistical Physics*, Secs. 4.1 and 4.3.4, University of Cambridge lecture notes, `https://www.davidtong.org/pdfs/teaching/statistical-physics/statphys4.pdf`.
[69] D. Baumann, "TASI Lectures on Inflation," arXiv:0907.5424 (2009), Secs. 12.3 and 13.
[70] R. Brustein and A. J. M. Medved, "A correspondence between strings in the Hagedorn phase and asymptotically de Sitter space," Phys. Rev. D **102**, 086002 (2020), arXiv:2005.09321, Eqs. (70)–(72).
[71] L. Boubekeur, "Theoretical bounds on the tensor-to-scalar ratio in the cosmic microwave background," arXiv:1208.0210 (2012), Sec. II.B.